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FILLING RIEMANNIAN MANIFOLDS

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0. Essential manifolds

A topological space K is said to be *aspherical* if the higher homotopy groups $\pi_i(K)$ for $i \geqslant 2$ vanish. This condition is equivalent to the contractibility of the universal covering $\tilde{K}$ of K in case K has the homotopy type of a cell complex. Then the homotopy type of K is uniquely determined by the fundamental group $\Pi = \pi_1(K)$, and K is called the *Eilenberg-MacLane* $K(\Pi, 1)$*-space*.

A closed (i.e., compact without boundary) connected manifold $V = V^n$ is said to be 1-*essential* (or, for brevity, *essential*) if for some map into an aspherical space, $f\colon V \to K$, the induced top dimensional homomorphism on homology does not vanish, i.e., $f_*[V] \neq 0$. Here $[V]$ stands for the *integral* fundamental class, i.e., $[V] \in H_n(V; \mathbf{Z}) \approx \mathbf{Z}$, in case the manifold V is oriented.

This $[V]$, for nonorientable manifolds V, denotes the fundamental $\mathbf{Z}_2$-class in the group $H_n(V; \mathbf{Z}_2) \approx \mathbf{Z}_2$.

Examples. Aspherical manifolds V are essential as the identity map f: $V \to V$ satisfies $f_*[V] \neq 0$. In particular, surfaces of positive genus are essential, as well as n-dimensional manifolds V, which admit metrics with nonpositive sectional curvatures. In fact the universal coverings of these manifolds V are homeomorphic to $\mathbf{R}^n$. The real projective space $P^n\mathbf{R}$ is essential for all $n = 1, \cdots$. Indeed the space $P^\infty\mathbf{R}$ is aspherical, and the inclusion map f: $P^n\mathbf{R} \to P^\infty\mathbf{R}$ gives the nonzero class, $0 \neq f_*[P^n\mathbf{R}] \in H_n(P^\infty\mathbf{R}; \mathbf{Z}_2) = \mathbf{Z}_2$, for all $n = 1, 2, \cdots$.

If V admits a map of nonzero degree onto an essential manifold V', then the manifold V is essential. In particular, connected sums $V = V' \# V''$ are essential for all closed manifolds V'', provided V' is essential.

0.1. Main isosystolic inequality. Let $\operatorname{sys}_1(V)$ denote the lower bound of the lengths of closed *noncontractible* curves γ in a Riemannian manifold V. (Compare §0.3.)

0.1.A. Theorem. *If V is a closed essential manifold of dimension n, then*

$$\operatorname{sys}_1(V) \leqslant \operatorname{const}_n (\operatorname{Vol} V)^{1/n}, \tag{0.1}$$

for some universal constant in the interval,

$$0 < \operatorname{const}_n < 6(n+1)n^n\sqrt{(n+1)!}\,.$$

In particular, V admits a closed geodesic γ such that

$$\operatorname{length} \gamma \leqslant \operatorname{const}_n (\operatorname{Vol} V)^{1/n}.$$

We prove this theorem in §§1.2 and 4.3 by first imbedding V into the space of functions $L^\infty(V)$ with the uniform norm (see §1.1) and then by analyzing quasi minimal $(n+1)$-dimensional chains which span (fill in) the cycle $[V]$ in $L^\infty(V)$. In fact, our main technical result is a generalization of *Federer-Fleming's isoperimetric inequality* (see [28] and §3) to infinite dimensional Banach spaces (see §4.2).

We introduce in §1 our key notion of the *filling radius* of V, which links together the isoperimetric problem and the inequality (0.1). The filling radius is accompanied by a whole spectrum of interesting geometric invariants of V which are discussed at various points in this paper.

Observe that the inequality (0.1) may fail for nonessential manifolds, no matter how large the fundamental group $\pi_1(V)$ is. In fact, if any manifold V' is multiplied by a *simply connected* manifold V'' of $\operatorname{Vol} V'' = \varepsilon$, then the product $V = V' \times V''$ has the same fundamental group as V' and also the same length of the shortest *noncontractible* geodesic, while $\operatorname{Vol} V = \varepsilon \operatorname{Vol} V' \to 0$ for $\varepsilon \to 0$.

(This does not exclude possible interesting relations between $\operatorname{Vol} V$ and inf length γ for contractible geodesics γ in nonessential manifolds V.)

0.2. An upper bound for the isosystolic constant. The isosystolic inequality (0.1) can be significantly improved for many essential manifolds V. Namely, we shall see in §6 that the constant const_n of the inequality (0, 1) tends to zero as the "topological complexity" of V goes to infinity.

Let a Riemannian manifold V be homeomorphic to a closed n-dimensional manifold V_0 of *negative sectional curvature* such that

$$\sup \text{curvature } V_0 \leqslant -1.$$

Then the total Riemannian volume of V_0 may serve as a measure of the topological complexity of $V_0 \approx V$. (Compare [74], [32].) We shall prove in §6.4 the following inequality with this volume $\operatorname{Vol} V_0$.

0.2.A. Theorem. *There exists a positive constant*, $\text{const} = \text{const}(n, \theta)$, *for every θ in the interval* $0 \leqslant \theta < 1$, *such that*

$$(\operatorname{Vol} V)(\operatorname{sys}_1 V)^n \geqslant \text{const}(\operatorname{Vol} V_0)^\theta. \tag{0.2}$$

Observe that the inequality (0.2) with $\theta = 0$ reduces to (0.1).

Notice that (0.2) may fail for $\theta = 1$. Indeed every manifold V_0 of *constant* negative curvature $= -1$ admits a sequence of finite d_i-sheeted coverings $V_0(i) \to V_0$, for $d_i \to \infty$ as $i \to \infty$, such that the shortest geodesic $\gamma(i)$ in $V(i)$ has length $\gamma(i) \to \infty$ for $i \to \infty$. (In fact, length $\gamma(i) \sim \log d_i$ for an appropriate sequence of coverings.) Thus the inequality (0.2) with $\theta = 1$ is violated by the manifolds $V_0(i)$ as $i \to \infty$.

0.3. Historical remarks and references. A classical result in geometry of numbers states that the length of the shortest closed geodesic γ in a flat torus T^n satisfies

$$\operatorname{sys}_1 T^n = \text{length}\,\gamma \leqslant \text{const}_n(\operatorname{Vol} T^n)^{1/n}.$$

It is easy to see that $\text{const}_2 = (2/\sqrt{3})^{1/2}$. The *extremal* 2-torus, for which $\operatorname{sys}_1 = [(2/\sqrt{3})\,\text{Area}]^{1/2}$, is the quotient of $\mathbf{R}^2$ by the *hexagonal* lattice, whose fundamental domain is a regular hexagon in $\mathbf{R}^2$.

Loewner proved that the 2-torus with an arbitrary (!) Riemannian metric also admits a noncontractible geodesic γ of length $l \leqslant [(2/\sqrt{3})\,\text{Area}]^{1/2}$. (See [64], [8].)

Loewner's proof is a straightforward application of the so-called *length-area method* (see [48]). The key ingredient of this method is the existence of a conformal homeomorphism of a torus T^2 with an arbitrary metric onto a *flat* torus T_0^2. If $\text{Area}\, T^2 = \text{Area}\, T_0^2$, then the *conformal* map $T^2 \to T_0^2$ increases the *lengths of the homotopy classes* of closed curves in T^2 (see §5.5), and thus the problem is reduced to the above case of flat tori.

Recall that the *length of a (homotopy) class* of curves by definition is the lower bound of the lengths of the curves in the class.

Pu pointed out (see [64]) that Loewner's method equally applies to Riemannian metrics on the real projective plane $\mathbf{R}P^2$. Pu's estimate for the shortest noncontractible geodesic is

$$l^2 \leqslant \pi/2 \text{ Area},$$

with the equality for the metrics of constant curvature on $\mathbf{R}P^2$.

Loewner's inequality was generalized by Accola [1] and Blatter [15] to surfaces V of genus $g \geqslant 1$,

$$(\mathrm{sys}_1 V)^2 \leqslant \mathrm{const}_g \mathrm{Area}\, V,$$

where $\mathrm{const}_1 = 2/\sqrt{3}$ and

$$\mathrm{const}_g \approx \frac{2g}{\pi e} \quad \text{for } g \to \infty.$$

(Observe that our inequality (0.2) is stronger for large $g \to \infty$. Namely $\mathrm{const}_g < g^{-\theta}$ for every fixed $\theta < 1$ and for large $g \to \infty$.)

These results were overhauled and brought into a general perspective in a series of papers by Berger (see [9], [10], [11]). Berger introduces the *k-dimensional systole* of a Riemannian manifold, $\mathrm{sys}_k V$, as the lower bound of the k-dimensional volumes of those k-dimensional subvarieties (possibly with singularities) in V which are not contractible (in V) to the $(k-1)$-skeleton of (some triangulation of) V. (Berger's original definition for $k > 1$ slightly differs from ours.) Berger has estimated these systoles for a variety of examples and proposed the following general question. What is the best constant const = const (topological type of V) for which the *isosystolic inequality*

$$\mathrm{sys}_k(V) \leqslant \mathrm{const}(\mathrm{Vol}\, V)^{k/n}$$

holds?

An easier question is to decide whether such a constant, const, exists at all for a given topological type of manifolds V. Our Theorem 0.1.A gives the positive answer to this question for the one-dimensional systoles of essential manifolds. The sharp constant in our inequality (0.1) of §0.1 is known only for surfaces homeomorphic to T^2 and to $\mathbf{R}P^2$ by the work of Loewner and Pu. In order to obtain the sharp constant for a more general manifold V of a given topological type one should determine (or at least guess) the *extremal* metric on V, for which the ratio $\mathrm{Vol}/(\mathrm{sys}_1)^n$, $n = \dim V$, assumes the least value. Unfortunately, the extremal metrics are not the "natural" metric one may expect. For example, no flat metric on the Klein bottle is extremal, as some small perturbations of the "square" flat Klein bottle (which has $(\mathrm{sys}_1)^2 = \mathrm{Area}$)

diminish the ratio $\text{Area}/(\text{sys}_1)^2$. This also happens to metrics of constant negative curvature on surfaces of genus $\geqslant 2$. (See [11].) However, every surface V_0 of constant negative curvature is extremal for an *asymptotic* (or stable) isosystolic inequality. Namely, let $N(V; l)$ denote the number of the homotopy classes of closed curves in V of length $< l$.

0.3.A. Theorem. *If the surfaces V and V_0 are homeomorphic and* Area $V =$ Area V_0, *then*

$$\lim_{l \to \infty} \left[\log N(V; l)/\log N(V_0; l)\right] \geqslant 1.$$

This is a special case of a more general result due to Katok [49]. Katok uses an ergodic version of the length-area method (see §5.5), and derives from his asymptotic isosystolic inequality the following remarkable corollary.

If the topological entropy of the geodesic flow on a closed surface V of negative curvature equals the metric entropy, then the surface V has constant curvature.

Katok's asymptotic inequality admits a generalization to manifolds of dimension $n \geqslant 3$. (See [32].) However, the "isosystolic constant" obtained in [32] is not sharp, and is yet unknown if hyperbolic manifolds (curvature $\equiv -1$) of dimension $\geqslant 3$ are extremal for the asymptotic isosystolic inequality.

There is a class of isosystolic inequalities which apply to manifolds with boundary. For example, *a lemma of Besikovič* gives the following lower bound for the volume of an arbitrary Riemannian metric on the n-dimensional cube in terms of the distances between the n pairs of opposite faces (see §7):

$$\text{Volume} \geqslant \prod_{i=1}^{n} \text{dist}_i. \tag{0.3}$$

Besikovič' lemma also applies to some closed manifolds (See [17], [37] and also §7.) Inequalities like (0.3) are sometimes called *inverse isoperimetric inequalities*. (See [17] where the reader will find an exhaustive account on various geometric inequalities.)

The volume of a Riemannian manifold V controls some other geometric invariants of V besides the systoles. Namely, the following remarkable *isembolic inequality* of Berger [12] gives a sharp upper bound for *the injectivity radius* of V:

$$\text{Inj Rad}\, V \leqslant \pi[\text{Vol}\, V/\text{Vol}\, S^n]^{1/n},$$

for the standard sphere S^n with the metric of constant curvature $+1$.

Furthermore, Hersch [44] has found the following sharp upper bound for *the first eigenvalue of the Laplace operator* on surfaces V diffeomorphic to the sphere S^2:

$$\lambda_1(V) \leqslant 2(\text{Area } S^2/\text{Area } V),$$

with the equality for $V = S^2$.

Hersch proves his inequality by using a conformal map $f\colon V \to S^2 \subset \mathbf{R}^3$, for which $\int_V f(v)\,dv = 0$. His method was generalized to surfaces of higher genus by Yang and Yau [77].

1. Filling radius

Let a closed connected n-dimensional manifold V be topologically imbedded into an arbitrary metric space, so that $V \subset X$. Denote by Fill Rad$(V \subset X)$ the infimum of those numbers $\varepsilon > 0$ for which V *bounds* in the ε-neighborhood $U_\varepsilon(V) \subset X$, that is the inclusion homomorphism $H_n(V) \to H_n(U_\varepsilon(V))$ vanishes. Here H_n may denote the singular homology with any given coefficients. In our applications below we use the coefficient group $\mathbf{Z}$ for orientable manifolds V and $\mathbf{Z}_2$ for nonorientable ones.

Examples. If V does not bound at all in X, for instance, if $V = X$, then Fill Rad $= \infty$.

The filling radius of a hypersurface $V^n \subset \mathbf{R}^{n+1}$ equals the radius of the largest ball in the region bounded by V^n. In particular, round spheres S^n in $\mathbf{R}^{n+1}$ (as well as S^n in $\mathbf{R}^q$ for $\mathbf{R}^q \supset \mathbf{R}^{n+1}$, $q \geqslant n+1$) have Fill Rad $=$ Rad S^n.

Next we define the *filling radius* of an abstract manifold V with a given metric by using an *isometric imbedding* of V into a Banach space.

Warning. Embeddings of Riemannian manifolds, $V \to W$, which preserve the *Riemannian* metric, may not be isometric. In fact, such embeddings are only *path-isometric*; they preserve the length of curves and therefore are distance-decreasing maps as $\operatorname{dist}(v_1, v_2) \overset{\text{def}}{=} \inf$ (the lengths of curves between v_1 and v_2 in V). However, such path-isometric maps may *strictly* decrease the distance. For example, no path-isometric map of a closed manifold V^n, $n > 0$, into $\mathbf{R}^q$ is isometric in our sense (see §9).

1.1. Imbedding $V \subset L^\infty(V)$. Let $L^\infty(V)$ be the Banach space of *bounded Borel functions f on V*, where

$$\|f\|_{L^\infty} \overset{\text{def}}{=} \sup_{w \in V} |f(w)|.$$

(As we allow discontinuous functions, the spaces $L^\infty(V)$ are isometric for all manifolds V of positive dimensions.)

Any distance function (that is a metric) on V imbeds V into $L^\infty(V)$,

$$v \mapsto f_v(w) = \mathrm{dist}(v, w).$$

The triangle inequality shows that this imbedding $V \subset L^\infty(V)$ is *isometric.*

Definition. The filling radius of a closed connected manifold V with a given metric is

$$\mathrm{Fill\,Rad}\, V = \mathrm{Fill\,Rad}(V \subset L^\infty(V)).$$

The space $L^\infty = L^\infty(V)$ is used for the following universal property: an arbitrary distance-decreasing map of a subspace of a metric space into L^∞,

$$Y \to L^\infty \text{ for } Y \subset X,$$

extends to a distance-decreasing map $X \to L^\infty$. Indeed, one may extend a map $y \to f_y(v) \in L^\infty = L^\infty(V)$ by the following map:

$$x \to f_x(v) = \inf_{y \in Y} \big(f_y(v) + \mathrm{dist}(x, y)\big),$$

for all $x \in X$. In particular, every distance-decreasing map $V_1 \to V_2$ extends to a distance-decreasing map $L^\infty(V_1) \to L^\infty(V_2)$. Hence *the filling radius decreases under distance-decreasing maps $V_1 \to V_2$ of degree one.*

Another corollary. *Isometric imbeddings into an arbitrary metric space $V \hookrightarrow X$ satisfy*

$$\mathrm{Fill\,Rad}(V \subset X) \geqslant \mathrm{Fill\,Rad}\, V.$$

Examples. The Riemannian sphere S^n of sectional curvature $+1$ isometrically imbedds as an equator into S^{n+1} and so

$$\mathrm{Fill\,Rad}\, S^n < \mathrm{Fill\,Rad}(S^n \subset S^{n+1}) = \pi/2.$$

In fact, Fill Rad $S^n = \frac{1}{2}\mathrm{arc}\,\cos(-\frac{1}{(n+1)})$, (see [50]). For instance, the filling radius of the circle of length 2π is $\pi/3$. The same circle $S^1 \subset \mathbf{R}^2$ but now with the (non-Riemannian!) metric induced from $\mathbf{R}^2$ has Fill Rad $S^1 = \sqrt{3}/2 <$ Fill Rad$(S^1 \subset \mathbf{R}^2) = 1$.

Also observe that all closed Riemannian manifolds V have

$$\mathrm{Fill\,Rad}\, V \leqslant \tfrac{1}{3}\,\mathrm{Diameter}\, V,$$

with the equality for real projective spaces of constant curvature (see [50]).

Furthermore, *the* Fill Rad *over* $\mathbf{Z}_2$ *of Cartesian products satisfy*

$$\mathrm{Fill\,Rad}(V_1 \times V_2) = \min(\mathrm{Fill\,Rad}\, V_1, \mathrm{Fill\,Rad}\, V_2).$$

Proof. If the "smallest" of the two manifolds, say $V_1 \subset L^\infty(V_1)$, bounds in some ε-neighborhood $U_\varepsilon(V_1) \subset L^\infty(V_1)$, then the product $V_1 \times V_2$ bounds in the product $U_\varepsilon(V_1) \times V_2$. This gives us the upper bound for Fill Rad$(V_1 \times V_2)$.

To get the lower bound we extend the projections

$$V_1 \times V_2 \to V_1 \text{ and } V_1 \times V_2 \to V_2$$

to some distance-decreasing maps

$$p_1\colon L^\infty(V_1 \times V_2) \to L^\infty(V_1) \text{ and } p_2\colon L^\infty(V_1 \times V_2) \to L^\infty(V_2).$$

If for some $\varepsilon > 0$ none of the manifolds V_1 and V_2 bounds in their respective ε-neighborhoods in $L^\infty(V_1)$ and $L^\infty(V_2)$, then the fundamental *cohomology* classes of V_1 and V_2 extend to some cohomology classes in these neighborhoods, say to the classes $\omega_1 \in H^m(U_\varepsilon(V_1))$ for $m = \dim V_1$, and to $\omega_2 \in H^n(U_\varepsilon(V_2))$ for $n = \dim V_2$. We take the pullbacks of these classes under the maps p_1 and p_2, and thus we get the classes in the ε-neighborhood $U_\varepsilon(V_1 \times V_2) \subset L^\infty(V_1 \times V_2)$,

$$\omega_1^* \in H^m(U_\varepsilon(V_1 \times V_2)) \text{ and } \omega_2^* \in H^n(U_\varepsilon(V_1 \times V_2)).$$

The cup product $\omega_1^* \cup \omega_2^*$ extends the fundamental class of the product $V_1 \times V_2$ to the ε-neighborhood $U_\varepsilon(V_1 \times V_2)$, and so the manifold $V_1 \times V_2$ does not bound in this ε-neighborhood.

1.2. An upper bound for the filling radius. We shall prove in §4.3 the following.

1.2.A. Main theorem. *Let V be a closed connected Riemannian manifold of dimension n. Then*

$$\text{(1.1)} \qquad \operatorname{Fill Rad} V \leqslant \text{const}'_n(\operatorname{Vol} V)^{1/n},$$

for some universal constant,

$$0 < \text{const}'_n < (n+1)n^n\sqrt{n!}\,.$$

This inequality (1.1) implies (0.1) of §0.1 with the following.

1.2.B. Lemma. *Let f be a continuous map of V to an aspherical space K. Suppose that all closed curves in V of length $\leqslant l$, for some $l > 0$, are sent by f to contractible curves in K. Let $P \subset L^\infty(V)$ be a simplicial polyhedron, which contains $V \subset L^\infty(V)$ as a subpolyhedron and is contained in the ε-neighborhood of V for some $\varepsilon > 0$,*

$$V \subset P \subset U_\varepsilon(V) \subset L^\infty(V).$$

If $l > 6\varepsilon$, then the map f extends to a continuous map $P \to K$. Furthermore, closed essential manifolds V have

$$\text{(1.2)} \qquad \operatorname{sys}_1 V \leqslant 6 \operatorname{Fill Rad} V.$$

Proof. As the space K is aspherical it suffices to extend the map f to the 2-skeleton of P.

We assume, subdividing the polyhedron P if necessary, that all simplices in P have diam $\Delta < \frac{1}{2}l - 2\varepsilon$. We send the vertices p_ν of P to some points v_ν in V for which

$$\operatorname{dist}(v_\nu, p_\nu) = \operatorname{dist}(p_\nu, V) < \varepsilon.$$

If a pair of points v_ν and v_μ in V comes from vertices of some simplex in P, say from p_ν and p_μ in Δ, then

$$\operatorname{dist}(v_\nu, v_\mu) < 2\varepsilon + \operatorname{diam} \Delta < \tfrac{1}{3}l,$$

and so this pair can be joined by a segment of length $< \frac{1}{3}l$ in V. We extend the map $p_\nu \mapsto v_\nu$ to a retraction of the 1-skeleton $P^1 \subset P$ to V by sending the edges of the complement $[p_\nu, p_\mu] \subset P \backslash V$ to these short segments in V. We additionally assume (subdivide P further if necessary) all edges $[p_\nu, p_\mu]$ in $V = P \cap V$ to be already short, of length $< \frac{1}{3}l$, or at least to be homotopic to such short segments. Then our retraction $P^1 \to V$ sends the boundary $\partial\Delta$ of each 2-simplex $\Delta \subset P$ to a curve of length $< l$ in V, or to a curve which can be homotoped to the length $< l$. The composition of the retraction $P^1 \to V$ with the map $f\colon V \to K$ extends this f to the 1-skeleton P^1 of P, and the boundary $\partial\Delta$ of every 2-simplex Δ goes to a contractible curve in K. Therefore this composition extends further to the 2-skeleton of P, and as K is aspherical any map extends from the 2-skeleton the whole polyhedron P.

1.2.C. The proof of the inequality (1.2). If the manifold $V \subset L^\infty(V)$ bounds in some ε-neighborhood $U_\varepsilon(V) \supset V$, then there exists by definition some singular chain c in $U_\varepsilon(V)$, whose boundary ∂c is contained in V and represents the fundamental class $[V]$ of V. Thus using a piecewise linear approximation of c one constructs a polyhedron P in $U_\varepsilon(V)$ such that V is contained in this P and the fundamental class $[V]$ vanishes under the inclusion homomorphism $H_n(V) \to H_n(P)$. Therefore no maps $f\colon V \to K$ for which $f_*[V] \neq 0$ extends to P, and *the length of the shortest noncontractible curve in the* (*essential*!) *manifold* V *is at most* 6 Fill Rad V.

This argument yields in fact the following generalization of the Theorem 0.1.A.

1.2.D. Theorem. *Let* $V = V^n$ *be a closed Riemannian manifold, and let* N_l *denote the normal subgroup in the fundamental group* $\pi_1(V)$ *which is generated by the homotopy classes of closed curves of length* $\leqslant l$. *Let* $f\colon V \to K(\Pi, 1)$ *be the classifying map for* $\Pi = \pi_1(V)/N_l$, *that is, f induces a surjective homomorphism* $\pi_1(V) \to \pi_1(K(\Pi, 1)) = \Pi$ *with kernel* N_l. *If* $l > \operatorname{const}_n(\operatorname{Vol} V)^{1/n}$ *for the constant* const_n *of Theorem* 0.1.*A*, *then the map f sends the fundamental class of V to zero, i.e.*, $f_*[V] = 0$.

1.2.D′. Example. *If V is homeomorphic to the connected sum of k tori T^n, then there exists k closed curves in V of length* $\leqslant \text{const}_n(\text{Vol}\, V)^{1/n}$ *whose homology classes generate a free abelian subgroup of rank k in the first homology group $H_1(V)$.*

Indeed, if some curves generate a subgroup $A \subset H_1(V)$ of rank $< k$, then by linear algebra there are some one-dimensional cohomology classes $\omega_1, \cdots, \omega_n$ in $H^1(V)$, whose cup-product equals the fundamental cohomology class $[V]^* \in H^n(V)$ and such that each class ω_i, $i = 1, \cdots, n$, vanishes on the subgroup $A \subset H_1(V)$. Using these ω_i one constructs a map f': $V \to T^n$ (compose Abel's map $V \to T^{nk}$ with a linear projection $T^{nk} \to T^n$), which sends A to zero in $H_1(T^n)$ and for which $f'_*[V] \neq 0$. Now the Theorem 1.2.D yields a curve in V of length $\leqslant \text{const}_n(\text{Vol}\, V)^{1/n}$ whose homology class *is not* in A (factor the map f' through the classifying map f: $V \to K(\pi_1(V)/N; 1)$ for the normal subgroup $N \subset \pi_1(V)$ generated by the curves in question). Therefore the group generated by *all* curves of length $\leqslant \text{const}_n(\text{Vol}\, V)^{1/n}$ has rank $\geqslant k$.

Remark. The inequality (1.2)

$$\text{sys}_1 \leqslant 6 \text{ Fill Rad}$$

is sharp. In fact, the real projective space of constant curvature $+1$ has

$$\text{sys}_1 \mathbf{R}P^n = 6 \text{ Fill Rad}\, \mathbf{R}P^n \pi/6,$$

(see [50]). Furthermore, a straightforward analysis shows that flat tori T^n also have

$$\text{sys}_1 T^n = 2 \text{ Inj Rad } T^n = 6 \text{ Fill Rad } T^n.$$

2. Filling volume

We define *the volume of a singular simplex* in an arbitrary metric space σ: $\Delta^{n+1} \to X$ as the lower bound of the total volumes of those Riemannian metrics on Δ^{n+1} for which the map σ is distance-decreasing. Then we also have the notion of the volume of a singular chain $c = \Sigma_i r_i \sigma_i$, namely,

$$\text{Vol}\, c = \sum_i |r_i| \text{Vol}\, \sigma_i,$$

where the coefficients r_i may be real numbers, integers of residues mod 2. In the latter case the "absolute value" $|r|$ is assumed zero for $r = 0$ and $|r| = 1$ for $r \neq 0$. Next for an n-dimensional singular *cycle* z in X we define the *filling volume* of z as the lower bound of the volumes of those $(n+1)$-dimensional chains c in X for which $\partial c = z$. The cycle z may be taken with integral, real or $\mathbf{Z}_2$ coefficients, and then one uses chains c with integral, real or $\mathbf{Z}_2$ coefficients respectively. Finally, we define the *filling volume of a closed submanifold* V in X as the lower bound of the filling volumes of n-dimensional cycles in $V \subset X$ which represent the fundamental class of V. For an abstract manifold V with a given metric we put

$$\text{Fill Vol}\, V = \text{Fil Vol}(V \subset L^\infty(V)).$$

2.1. Filling volumes of hypersurfaces. Let W be a compact Riemannian manifold with boundary. Then the boundary ∂W has, with the induced (non-Riemannian) metric,

$$\text{Fill Vol}\, \partial W \leqslant \text{Vol}\, W. \tag{2.1}$$

Indeed, the isometric embedding $\partial W \hookrightarrow L^\infty(\partial W)$ extends to a distance-decreasing map $W \to L^\infty(\partial W)$.

Furthermore, *if the manifold W is Riemannian flat as well as simply connected, then*

$$\text{Fill Vol}\, \partial W = \text{Vol}\, W.$$

Proof. The manifold W admits a locally isometric map into $\mathbf{R}^{n+1}$ for $n+1 = \dim W$, and this map is distance-decreasing (nonstrictly) on the boundary ∂W. It suffices to show that this map on the boundary extends to a *volume-decreasing* map of any chain which spans (fills in) the manifold ∂W in $L^\infty(\partial W)$. In fact, we shall prove in §4.1 the following more general result.

2.1.A. Proposition. *Take a subspace Y in an arbitrary metric space X, and let $f\colon Y \to \mathbf{R}^{n+1}$ be a distance-decreasing map. Then this map f extends to a Lipschitz map $F\colon X \to \mathbf{R}^{n+1}$ for which*

$$\text{dist}(F(x_1), F(x_2)) \leqslant \sqrt{n+1}\, \text{dist}(x_1, x_2), \quad \textit{for } x_1, x_2 \in X,$$

and which decreases the volumes of all $(n+1)$-chains in X.

The only known examples of equality in (2.1) come from the flat manifolds above and also from more general Riemannian domains (which may be ramified if singular metrics are allowed) over $\mathbf{R}^{n+1}$. However, the inequality (2.1) can not be, in general, improved as the following consideration shows.

2.2. The filling volume of the boundary ∂W of W. Take a closed n-dimensional manifold V with a metric dist_0, and take an $(n+1)$-dimensional manifold W with boundary $\partial W = V$. Consider all those Riemannian metrics g on W for which the corresponding distance functions dist_g satisfy on the boundary $\partial W = V$

$$\text{dist}_g \mid \partial W \geqslant \text{dist}_0, \tag{2.2}$$

that is,

$$\text{dist}_g(v_1, v_2) \geqslant \text{dist}_0(v_1, v_2) \quad \text{for all } v_1 \text{ and } v_2 \text{ in } V.$$

Denote by $M = M(V, W)$ the lower bound of the total volumes of the metrics g:

$$M = \inf_g \mathrm{Vol}(W, g).$$

This number M is the minimal volume of the Riemannian manifolds of a fixed topological type (homeomorphic to W), which span (fill in) $V = (V, \mathrm{dist}_0)$.

If the manifold V is oriented, then W also is assumed to be oriented. Furthermore, we extend the definition of $M(V, W)$ to noncompact manifolds W (yet with boundary $\partial W = V$) by allowing only those competing Riemannian metrics g on W for which the metrics d_g on W are complete (otherwise the minimal volume $M(V, W)$ vanishes for noncompact manifolds W).

In fact, this minimal volume $M(V, W)$ depends only on $V = (V, \mathrm{dist}_0)$ but not on the topology of W, provided V is connected and oriented and $\dim V \geqslant 2$. Moreover, we shall prove in Appendix 2 the following.

2.2.A. Proposition. *If V is a connected oriented manifold of dimension* $\geqslant 2$, *then*

$$M = \mathrm{Fill\,Vol}\, V. \tag{2.3}$$

This equality leads to an alternative definition of the filling volume even for those manifolds V which bound no *compact* manifold W. Indeed, every V bounds the product $V \times [0, \infty)$.

This result also shows that the filling volume of the product of two manifolds V_1 and V_2 does not exceed

$$\min[(\mathrm{Vol}\, V_1)\mathrm{Fill\,Vol}\, V_2, (\mathrm{Vol}\, V_2)\mathrm{Fill\,Vol}\, V_1].$$

However, the equality need not hold. For example, the inequality is *strict* for the product of two unit circles $S^1 \subset \mathbf{R}^2$ with the induced (non-Riemannian) metrics.

I do not know if there is such an example of a *Riemannian* manifold. In fact, I do not know the explicit value of the filling volume of any single Riemannian manifold. A natural *conjecture* for the spheres of constant curvature is

$$\mathrm{Fill\,Vol}\, S^n = \tfrac{1}{2}\mathrm{Vol}\, S^{n+1}.$$

2.2.B. Counterexamples. (1) Take a connected bounded domain W in $\mathbf{R}^{n+1}$ whose boundary ∂W has two components, i.e., $\partial W = V_1 \cup V_2$. These manifolds V_1 and V_2 bound domains, say W_1 and W_2 in $\mathbf{R}^{n+1}$, such that $W = W_1 \backslash W_2$ (or $W = W_2 \backslash W_1$ if V_1 is inside V_2). We have according to Propositions 2.1.A and 2.2.A

$$M = M(\partial W, W) = \mathrm{Vol}\, W,$$

as well as

$$M_i = M_i(V_i, W_i) = \operatorname{Vol} W_i \quad \text{for } i = 1, 2,$$

and so M is *strictly* less than

$$M_1 + M_2 = M(\partial W, W_1 \cup W_2).$$

Therefore *disconnected* manifolds $V = \partial W$ may violate the equality (2.3).

(2) Now let W be a compact connected surface of genus $g > 0$ with boundary $\partial W = S^1$. Take a flat Riemannian metric g_0 on W induced by some immersion $W \to \mathbf{R}^2$. Then every surface W' of genus $q' < q$ which spans the manifold $\partial W = S^1$ with the induced metric $\operatorname{dist}_{g_0} | \partial W$ has

$$(\operatorname{Vol} W') - \varepsilon \geqslant \operatorname{Vol} W = \operatorname{Fill\,Vol} \partial W,$$

for some fixed number $\varepsilon > 0$, and so this manifold $S^1 = (\partial W, \operatorname{dist}_{g_0} \partial W)$ violates (2.3). Moreover, if one allows degenerate metrics on S^1, for example the "metric" induced by a map f of S^1 into a closed surface X of genus q, such that this f lifts in the universal covering $\tilde{X} \to X$ to a homeomorphism of S^1 onto the boundary of some fundamental domain, then one gets

$$M(S^1, W) = 0 \qquad \text{for genus}(W) \geqslant q,$$
$$M(S^1, W) \geqslant \varepsilon > 0 \quad \text{for genus}(W) < q.$$

(3) Take the degenerate "metric" on the sphere S^n induced from the real projective space P^n by the double covering map $S^n \to P^n$. The *non-oriented* filling volume of S^n with this "metric" is zero. However, the *oriented* filling volume is nonzero for n odd. Yet it is zero for n even. Therefore this "metric" on S^n (or rather an actual metric obtained by an arbitrary small perturbation) extends to a complete metric on the product $S^n \times [0, \infty)$ with an arbitrary small volume for n even but not for n odd.

2.3. Isoperimetric inequality (*see* §4.2). *The filling volume of every n-dimensional cycle z (with real, integral or $\mathbf{Z}_2$-coefficients) in an arbitrary space L^∞ satisfies*

$$\operatorname{Fill\,Vol}(z) \leqslant C_n (\operatorname{Vol} z)^{(n+1)/n}, \tag{2.4}$$

for some universal constant in the interval $0 < C_n < n^n \sqrt{(n+1)!}$. *In particular, all closed connected Riemannian manifold V have*

$$\operatorname{Fill\,Vol}(V) \leqslant C_n (\operatorname{Vol} V)^{(n+1)/n}. \tag{2.5}$$

The inequality (2.5), together with the Propositin 2.2.A, implies the following.

2.3.A. Corollary. *An arbitrary Riemannian metric g_0 on a closed manifold V can be extended to a complete Riemannian metric g on the infinite cylinder*

$W = V \times [0, \infty)$ *such that*

$$\operatorname{dist}_g \mid V \times 0 = \operatorname{dist}_{g_0},$$

(2.6) $$\operatorname{Vol}(W, g) \leq C_n(\operatorname{Vol}(V, g_0))^{(n+1)/n}.$$

The case of $V = S^1$. Proposition 2.2.A does not formally apply to S^1, but Corollary 2.3.A is obvious for S^1 anyway.

Remarks. The inequality

$$\operatorname{Fill\,Vol} \leq \operatorname{const}_N(\operatorname{Vol})^{(n+1)/n}$$

for n-dimensional cycles in $L = \mathbf{R}^N$ is due to Federer and Fleming [28]. This result was sharpened by Michael and Simon [58] who proved the isoperimetric inequality (2.4) for $L = \mathbf{R}^N$ with a constant which depends on $n = \dim(\text{cycle})$ rather than on the dimension N of the ambient space. This is equivalent to (2.4) in the *infinite* dimensional Hilbert space.

Our proof of (2.4) in §§3.3 and 4.2 closely follows the original elementary argument of Federer and Fleming. We do not rely, as Michael and Simon do, on the theory of minimal varieties. Observe that the inequality of Michael and Simon also applies to *locally* minimizing subvarieties in $\mathbf{R}^N$, while the inequality (2.4) needs a nontrivial modification in order to apply to *locally* minimizing subvarieties in a general Banach space. In fact, by our choice of "Vol" there even exist closed (!) locally minimizing submanifolds in some Banach spaces. For example, Flat tori $T^n \subset L^\infty(T^n)$ are locally minimizing as it follows from Proposition 2.1.A.

2.4. On the relation between the filling volume and the filling radius. Let us assume that the inequality (1.1) does hold with some *universal constant* const'_n for all manifolds V of dimension n. Then we claim the relation

(2.6) $$\operatorname{const}'_n \leq C'_n = (n+1)C_n,$$

for the isoperimetric constant C_n of the inequality (2.5). That is, we claim (1.1) to hold with const'_n replaced by the *new* constant C'_n.

To see this we take an arbitrary manifold V, span it by a cylinder $W = V \times [0, \infty)$, for which

(2.7) $$\operatorname{Vol} W \leq C'_n(\operatorname{Vol} V)^{(n+1)/n},$$

and then consider the levels of the distance function

$$d(w) = \operatorname{dist}(w, | V \times 0),$$

which are

$$V_r = d^{-1}(r) \subset W \text{ for all } r \geq 0.$$

Assume for the moment these levels to be smooth submanifolds in W, and observe the obvious inequality for the filling radii R_r of the manifolds V_r with the induced Riemannian metrics,

$$R_r \geqslant R_0 - r \quad \text{for all } r \geqslant 0. \tag{2.8}$$

The inequality (1.1) for $R_r = \text{Full Rad } V_r$ yields

$$\text{const}'_n(\text{Vol } V_r)^{1/n} \geqslant R_0 - r. \tag{2.9}$$

We invoke the *coarea formula*,

$$\int_0^{R_0} (\text{Vol } V_r)\, dr = \text{Vol } d^{-1}[0, R_0] < \text{Vol } W,$$

which together with the inequality (2.9) gives

$$\int_0^{R_0} (R_0 - r)^n \leqslant (\text{const}'_n)^n \text{Vol } W,$$

that is,

$$R_0^{n+1} \leqslant (n+1)(\text{const}'_n)^n \text{Vol } W. \tag{2.10}$$

This inequality (2.10) says, in fact, that

$$(\text{Fill Rad } V)^{n+1} \leqslant (n+1)(\text{const}'_n)^n \text{ Fill Vol } V.$$

Now we apply (2.7) and obtain

$$R_0 \leqslant (C'_n)^{1/(n+1)}(\text{const}'_n)^{n/(n+1)}\text{Vol } V. \tag{2.11}$$

The inequality (2.11) improves (1.1) as long as $\text{const}'_n > C'_n = (n+1)C_n$, and so the manifold $V = V_0$ does satisfy (1.1) with const'_n replaced by the constant C'_n.

Finally, we remove the regularity assumption on the levels V_r by taking a smooth approximation of the distance function $d(w)$.

Warning. The above argument *does not* prove that the inequality

$$\text{Fill Rad} \leqslant C'_n(\text{Vol})^{1/n}$$

follows from (2.7), as the constant const'_n of (1.1) might apriory be infinite. However, we shall refine this conditional argument in §4.3 in order to make it free of the assumption $\text{const}'_n < \infty$. Only then we shall prove the implication (2.6).

3. Filling inequalities for submanifolds V in $\mathbf{R}^N$

3.1. The method of Federer-Fleming. Let V be an arbitrary n-dimensional submanifold in the Euclidean space $\mathbf{R}^N$.

3.1.A. Proposition. *There exists a continuous map f of V into an $(n-1)$-dimensional subpolyhedron K^{n-1} in $\mathbf{R}^N$, such that the Euclidean distance between f and the identity map* Id: $V \to V \subset \mathbf{R}^N$ *satisfies*

$$\operatorname{dist}(v, f(v)) \leqslant C_N(\operatorname{Vol} V)^{1/n},$$

for all $v \in V$ and for some constant $C_N \leqslant \frac{1}{2}\sqrt{N}(N!/(n!(N-n)!))^{1/n}$.

Proof. We divide the space $\mathbf{R}^N$ into the unit N-dimensional cubes which are the fundamental domains of the lattice in $\mathbf{R}^N$ spanned by a fixed system of N orthonormal vectors $e_i \in \mathbf{R}^N$, $i = 1, \cdots, N$. Observe that the i-skeleton K^i of this subdivision of $\mathbf{R}^N$ is the *grid* of integral translates of the i-dimensional coordinate subspaces $\mathbf{R}^i_\nu$ for $\nu = 1, \cdots, N!/(i!(N-i)!)$.

Let us project all cubes radially from the centers to their respective boundaries, and let us denote the resulting (discontinuous) map by P_{N-1}: $\mathbf{R}^N \to K^{N-1}$. Next we apply these radial projections to the $(N-1)$-faces of our cubes, then to $(N-2)$-faces and so on. Thus we obtain some maps P_i: $\mathbf{R}^N \to K^i$ for all $i = N-1, N-2, \cdots, 0$. Each map P_i is *continuous outside the dual grid* of $(N-i-1)$-dimensional subspaces which is the translate of K^{N-i-1} by the vector $\frac{1}{2}\Sigma_{i=1}^N e_i$.

Denote by $Q = Q(V)$ the sum of the volumes of the *images* of the normal projections of the manifold V to the coordinate subspaces $\mathbf{R}^n_\nu$ for $\nu = 1, \cdots, N!/(n!(N-n)!)$. If $Q < 1$, then some parallel translate V' of V does not intersect the dual $(N-n)$-grid, and so the map P_{n-1} is *continuous* on V'. As $Q < N!/(n!(N-n)!)\operatorname{Vol} V$ and $\operatorname{dist}(P_{n-1}, \mathrm{Id}) \leqslant \frac{1}{2}\sqrt{N}$, this map P_{n-1} satisfies the required relation on V' in case $\operatorname{Vol}(V') = \operatorname{Vol}(V) = (n!(N-n)!)/N!$. Finally, any manifold can be scaled to have volume $= (n!(N-n)!)/N!$, and the proof is completed.

3.1.A′. Corollary. *The filling radius of every closed submanifold V in $\mathbf{R}^N$ satisfies*

$$\operatorname{FillRad}(V \subset \mathbf{R}^N) \leqslant C_N(\operatorname{Vol} V)^{1/n}. \tag{3.1}$$

Proof. The manifold V is spanned by the cylinder of the map f.

This corollary is sharpened and generalized in §4.3. However, Proposition 3.1.A and its proof carry some additional information which is not contained in the inequality (3.1). Namely, Proposition 3.1.A estimates another invariant $\operatorname{Rad}_k(V \subset \mathbf{R}^N)$ which is defined as the lower bound of those numbers $\varepsilon > 0$ for which there exists a continuous map f of V into some k-dimensional subpolyhedron of $\mathbf{R}^N$, such that $\operatorname{dist}(f, \mathrm{Id}) \leqslant \varepsilon$. Then Proposition 3.1.A claims the inequality

$$\operatorname{Rad}_{n-1}(V \subset \mathbf{R}^N) \leqslant C_N(\operatorname{Vol} V)^{1/n} \tag{3.2}$$

for all (not necessarily compact) n-dimensional submanifolds V in $\mathbf{R}^N$.

3.1.A″. Examples. (a) If V is *connected*, then $\mathrm{Rad}_0(V \subset \mathbf{R}^N)$ equals the radius of the *minimal* ball in $\mathbf{R}^N$ which contains V.

(b) If the manifold V is *simply connected*, then it is contractible in the ε-neighborhood $U_\varepsilon(V) \subset \mathbf{R}^N$ for $\varepsilon = \mathrm{Rad}_1(V \subset \mathbf{R}^N)$. Indeed, if $\pi_1(V) = 0$, then the map $f\colon V \to K^1$ is contractible.

In particular, *if the manifold V is homeomorphic to S^2, then it is contractible in the neighborhood $U_\varepsilon(V) \subset \mathbf{R}^N$ for $\varepsilon \leqslant C_N(\mathrm{Vol}\, V)^{1/2}$*, as it follows from (3.2). This contractibility property may fail for higher dimensional spheres. For example, the Hopf map $S^3 \to S^2 \subset \mathbf{R}^3$ gives a 3-dimensional "sphere" in $\mathbf{R}^3$ of zero volume, which is contractible in no ε-neighborhood for small $\varepsilon > 0$.

(c) *Let V be an n-dimensional submanifold with boundary in $\mathbf{R}^n$. Then* $\mathrm{Rad}_{n-1}(V \subset \mathbf{R}^n)$ *equals the radius of the largest Euclidean ball inside V.*

Proof. Denote by $V - \varepsilon$ the largest subset in V whose ε-neighborhood also is contained in V. Clearly, this set $V - \varepsilon$ is contained in the image of any continuous map $f\colon V \to \mathbf{R}^N$, for which $\mathrm{dist}(f, \mathrm{Id}) \leqslant \varepsilon$. The resulting inequality $\mathrm{Vol}(\mathrm{Image}_f) \geqslant \mathrm{Vol}(V - \varepsilon)$ gives us the desired lower bound for $\mathrm{Rad}_{n-1}(V \subset \mathbf{R}^n)$.

Next we consider the cut locus $\mathrm{Cut} \subset V$ of V relative to the boundary ∂V. Then the normal map $f\colon V \to \mathrm{Cut}$ gives us the upper bound for $\mathrm{Rad}_{n-1}(V \subset \mathbf{R}^n)$, since $\dim \mathrm{Cut} \leqslant n - 1$.

Remark. The classical isoperimetric inequality implies

$$\mathrm{Vol}(V - \varepsilon) \leqslant (1 - \varepsilon/R)\mathrm{Vol}\, V, \quad \text{for } \varepsilon \leqslant \mathrm{Rad}_{n-1}(V \subset \mathbf{R}^n),$$

where R denotes the radius of the Euclidean n-ball of volume $= \mathrm{Vol}\, V$. Then the (continuous!) normal map f_ε of V onto the union $\mathrm{Cut} \cup (V - \varepsilon)$ satisfies for all positive $\varepsilon \leqslant \mathrm{Rad}_{n-1}(V \subset \mathbf{R}^n)$,

$$\mathrm{dist}(f_\varepsilon, \mathrm{Id}) \leqslant \varepsilon,$$
$$\mathrm{Vol}(\mathrm{Image}\, f_\varepsilon) \leqslant (1 - \varepsilon/R)\mathrm{Vol}\, V.$$

The existence of maps f_ε with these properties is, in fact, equivalent to the classical isoperimetric inequality.

Question. Can one replace the constant C_N in the inequality (3.2) by the constant which depends only on $n = \dim V$?

The proof of the Proposition 3.1.A admits several improvements. For example, one may average over all orthogonal frames $(e_1, \cdots, e_N)$ in $\mathbf{R}^N$ thus diminishing the factor $\sqrt{N}$ of C_N. (One could avoid the trouble by using the l_∞ norm in $\mathbf{R}^N$, that is, $\|x\|_{l_\infty} = \max_{1 \leqslant i \leqslant N} |x_i|$, instead of the Euclidean l_2-norm .) Then one could use the *independence* of translations in the directions of *mutually orthogonal* subspaces $\mathbf{R}^n_\nu$, thus diminishing the second factor of C_N. Finally, one might seek for a *sharp* inequality like (3.2) but with the quantity

$Q(V)$ in place of Vol V. Unfortunately, none of these improvements reduce C_N to C_n.

3.2. The isoperimetric inequality of Federer-Fleming. *An arbitrary n-dimensional cycle z in* $\mathbf{R}^N$ *satisfies*

$$\text{Fill Vol}(z) \leqslant C_N(\text{Vol } z)^{(n+1)/n},$$

for some universal constant C_N.

PROOF. Let K' denote the dual $(N-n-1)$-grid (see the proof of Proposition 3.1.A). Then the norm of the differential of the projection P_n: $\mathbf{R}^N \to K^n$ satisfies

$$\|Df(x)\| \leqslant A_N[\text{dist}(x, K')]^{-1},$$

for some constant A_N. Indeed, the map P_n is piecewise projective; it projects some simplices with integer and half-integer vertices to some lower dimensional faces. As the integral of the function $\delta(x) = (\text{dist}(x, K'))^{-n}$ over any unit N-dimensional cube is *finite*, there exists a translate z' of the cycle z such that the integral of $\delta(x)$ over z' satisfies

$$\int_{z'} \delta(x)\, dz' \leqslant A'_N \text{Vol } z,$$

where A'_N is a universal constant. Therefore the projected cycle $z'' = P_n(z')$ has

$$\text{Vol } z'' \leqslant A'_N \text{Vol } z,$$

and the cylinder of the map P_n satisfies

$$\text{Vol(cylinder)} \leqslant \tfrac{1}{2}\sqrt{N} A'_N \text{Vol } z.$$

Since all n-dimensional cubes of the complex K^n have unit volume, the number of those cubes which are totally covered by the cycle z'' is at most Vol z''. It follows that z'' is homologous in K^n to a cubic cycle z''' which consists of at most Vol z'' unit cubes. The chain c_0 in K^n for which $\partial c_0 = z''' - z''$ has zero $(n+1)$-dimensional volume.

Finally, the diameter of (the support of) z''' is at most $\sqrt{n}\,$Vol z'', and so z''' bounds a cone for which

$$\text{Vol(cone)} \leqslant \frac{1}{n+1}(\text{diam } z''')\text{Vol } z''' \leqslant A'_N \frac{n}{n+1}(\text{Vol } z)^2.$$

Now the chain $c = \text{Cylinder} + c_0 + \text{cone}$ which spans the cycle z' has Vol(c) $\leqslant \text{const}_N(\text{Vol } z + (\text{Vol } z)^2)$.

This implies the isoperimetric inequality in case Vol $z = 1$. If Vol $z \neq 1$, we scale the cycle z to the unit volume. q.e.d.

This proof is due to Federer-Fleming [28]. Also see [17], [16].

3.2.A. An important remark. The Federer-Fleming construction delivers a filling chain c which is contained in the ε-neighborhood U_ε of (the support of) z for some $\varepsilon \leqslant \text{const}_N(\text{Vol } z)^{1/n}$. Thus we get a bound on the filling radius of the cycle z as well as on its filling volume. Moreover, we get the following sharpened isoperimetric inequality:

$$\text{Fill Vol}(z \subset U_\varepsilon) \leqslant C_N(\text{Vol } z)^{(n+1)/n}.$$

3.3. The Federer-Fleming inequality with a constant C_n for $n = \dim V$. We shall prove in this section the inequality

$$\text{Fill Vol}(V \subset \mathbf{R}^N) \leqslant C_n(\text{Vol } V)^{(n+1)/n}, \tag{3.3}$$

for closed submanifolds V in $\mathbf{R}^N$. This result is due to Michael and Simon, who derive (3.4) from their more powerful isoperimetric inequality for minimal subvarieties W in $\mathbf{R}^N$ which span V. Our proof below is more elementary and generalizes to submanifolds in Banach spaces. This is crucial for our ultimate purpose of estimating the filling radius of a (non-embedded!) Riemannian manifold.

Our proof is based on the two following elementary facts.

(1) **The cone inequality:**

$$\text{Fill Vol}(V \subset \mathbf{R}^N) \leqslant \frac{1}{n+1}(\text{Diam } V)\text{Vol } V.$$

In fact, V bounds the cone from a point $v_0 \in V$ over V, and

$$\text{Vol(cone)} \leqslant \frac{1}{n+1}\int_V \text{dist}(v, v_0)\, dv \leqslant \frac{1}{n+1}(\text{Diam } V)\text{Vol } V.$$

The cone inequality also holds for an arbitrary cycle z in $\mathbf{R}^N$, where we define

$$\text{Diam } z = \text{Diam(support } z).$$

(2) **The coarea inequality.** Let $d(x)$, $x \in \mathbf{R}^N$, denote the distance to a subset $H \subset \mathbf{R}^N$:

$$d(x) = \text{dist}(x, H).$$

We intersect V with the levels of the function $d(x)$, we put

$$a(t) = V \cap d^{-1}(t), \quad b(t) = V \cap d^{-1}[0, t].$$

The $(n-1)$-dimensional volumes

$$\alpha(t) = \text{Vol } a(t),\ t \in [0, \infty),$$

are related to the n-dimensional volumes

$$\beta(t) = \text{Vol } b(t)$$

by the *coarea inequality*

$$\beta(t) \geqslant \int_0^T \alpha(t)\, dt, \quad \text{for all } T > 0.$$

This inequality also applies to an arbitrary n-dimensional cycle z in $\mathbf{R}^N$. Observe that the intersections $a(t) = z \cap d^{-1}(t)$ are $(n-1)$-dimensional cycles for almost all $t \in (0, \infty)$, while the intersections $b(t) = z \cap d^{-1}[0, t]$ are n-dimensional chains such that $\partial b(t) = a(t)$ for almost all t. There may appear some minor regularity problems depending on a particular class of singular chains in question. (These may be Lipschitz chains, piecewise smooth chains, piecewise linear chains etc.) However, all such problems disappear with an obvious approximation of the function $d(x)$ and (or) the cycle z by more regular objects.

We prove (3.3) by (1) and (2) roughly as follows. We denote by $C_{\max} = C(\mathbf{R}^N)$ the upper bound of the functional Fill Vol$/(\text{Vol})^{(n+1)/n}$ over all n-dimensional submanifolds (or cycles) in $\mathbf{R}^N$. The Federer-Fleming inequality implies

$$C_{\max} \leqslant C_N < \infty.$$

We must show that, in fact, $C_{\max} \leqslant C_n$. Assume for the moment the existence of an *extremal* submanifold V in $\mathbf{R}^N$, for which

$$\text{Fill Vol}\,(V \subset \mathbf{R}^N) = C_{\max} \operatorname{Vol} V.$$

(It is not hard to show that an extremal "manifold with singularities" does exist, but we shall need this fact for our proof.) If the diameter of this extremal V abides

$$\operatorname{Diam} V \leqslant D_n (\operatorname{Vol} V)^{1/n},$$

for some universal constant D_n, then we obtain the proof with the cone inequality. Otherwise, the manifold V has "very large" diameter, and then we decompose V into a sum of two "submanifolds" (or rather of two cycles):

$$V = V_1 + V_2,$$

such that

$$(\operatorname{Vol} V_1)^{(n+1)/n} + (\operatorname{Vol} V_2)^{(n+1)/n} < (\operatorname{Vol} V)^{(n+1)/n}.$$

The existence of such a decomposition clearly contradicts the extremality of the manifold V, and thus we exclude the possibility of

$$\operatorname{Diam} V > D_n (\operatorname{Vol} V)^{1/n}.$$

To decompose V we consider the distance function $d(x) = \operatorname{dist}(x, H)$ to an appropriate subset $H \in \mathbf{R}^N$, and by the coarea formula we find a section

$a(t) = V \cap d^{-1}(t)$ of "small" $(n-1)$-dimensional volume. Then assuming by induction the inequality (3.3) for n replaced by $n-1$ we span the cycle $a(t)$ by a chain $\tilde{b}(t)$ of small n-dimensional volume $\tilde{\beta}(t)$. Thus we decompose V into the sum

$$V = [b(t) - \tilde{b}(t)] + [b'(t) + \tilde{b}(t)],$$

for $b(t) = V \cap d^{-1}[0, t]$ and $b'(t) = V - b(t) = V \cap d^{-1}[t, \infty)$. A straightforward calculation (see the analytic lemmas below) shows that this decomposition (for some choice of $t \in [0, \infty)$) does diminish the "weighted volume" $(\mathrm{Vol})^{(n+1)/n}$. Moreover, by such a decomposition we shall prove an isoperimetric inequality for an arbitrary Riemannian manifold $X \supset V$, which satisfies an appropriate "cone inequality" (see Corollary 3.4.C and Appendix 2).

Analytic lemmas

(A_1) *Let two positive functions $\alpha(t)$ and $\beta(t)$ in the interval $t \in [0, T_0]$ satisfy*

$$\beta(T) \geq \int_0^T \alpha(t)\, dt, \quad \text{for all } T \in [0, T_0],$$

and

$$\beta(t) \leq c\alpha(t)^{n/(n-1)}, \quad t \in [0, T_0],$$

for some $n \geq 2$ and a positive constant c. Then

$$\beta(t) \geq t^n / (c^{n-1} n^n),$$

for all $t \in [0, T_0]$.

The proof is obvious by observing that the equality holds for

$$\alpha(t) = \frac{t^{n-1}}{c^{n-1} n^{n-1}}, \quad \beta(t) = \int_0^t \alpha(\tau)\, d\tau = \frac{t^n}{c^{n-1} n^n}.$$

(A_2) *Let β_0 and β_1 be positive numbers such that*

$$\delta\beta_1 \leq 2\beta_0 \leq \beta_1,$$

for some δ in the interval $0 < \delta \leq 1$. Then

$$(\beta_0 + \varepsilon\beta_0)^{(n+1)/n} + (\beta_1 - \beta_0 + \varepsilon\beta_0)^{(n+1)/n} \leq (1 - \delta')\beta_1^{(n+1)/n},$$

where ε is an arbitrary number in the interval $\varepsilon \in [0, \frac{1}{2}n]$, and δ' is a positive constant, which depends only on δ, $\delta' = \delta'(\delta) > 0$.

Proof. This inequality for $\beta_1 = 2\beta_0$ amounts to the obvious relation

$$\left(1 + \frac{1}{2n}\right)^{n+1} < 2.$$

The case $\beta_1 > 2\beta_0$ then follows from the convexity of the function $x^{(n+1)/n}$.

(A_3) *Let some positive functions* $\alpha(t)$, $\beta(t)$ *and* $\tilde{\beta}(t)$ *satisfy in the interval* $t \in [0, T_1]$,

(i) $\beta(T) \geq \int_0^T \alpha(t)\,dt$,

(ii) $\beta(t)$ *is monotone increasing in the interval* $[0, T_1]$,

(iii) $\tilde{\beta}(t) \leq C_{n-1}[\alpha(t)]^{n/(n-1)}$, *for some constant* $C_{n-1} > 0$.

Then one of the two following alternatives takes place:

(1) *There is a value* $t_0 \in [0, T_1)$ *such that*

$$[\beta(t_0) + \tilde{\beta}(t_0)]^{(n+1)/n} + [\beta(T_1) - \beta(t_0) + \tilde{\beta}(t_0)]^{(n+1)/n} \leq (1 - \delta')\beta(T_1)^{(n+1)/n}, \tag{3.4}$$

for some positive number δ' *which depends only on* $\delta = \beta(0)/\beta(T_1)$ *such that* $\delta' \to 0$ *for* $\delta \to 0$.

(2) *There is a subinterval* $[0, T_0]$ *in the interval* $[0, T_1]$ *such that*

$$\beta(T_0) \geq \tfrac{1}{2}\beta(T_1), \quad T_0 \leq \tfrac{1}{2}D_n[\beta(T_1)]^{1/n},$$

for $D_n = 2^{2-2/n}n^{(2n-1)/n}C_{n-1}^{(n-1)/n} < 4n^2C_{n-1}^{(n-1)/n}$.

Proof. Let T_0 be the upper bound of those values $t \in [0, T_1]$, for which $\beta(t) \leq \frac{1}{2}\beta(T_1)$. If the inequality (3.1) fails for all $t \in [0, T_0)$, then by Lemma A_2,

$$\tilde{\beta}(t) \geq \varepsilon, \quad \text{for } t \in [0, T_0), \varepsilon = 1/(2n).$$

Therefore

$$\varepsilon\beta(t) \leq C_{n-1}\alpha(t)^{1/(n-1)}, \quad \text{for } t \in [0, T_0),$$

and Lemma (A_1) with $c = \varepsilon^{-1}C_{n-1}$ implies

$$\beta(t) \geq \varepsilon^{n-1}t^n/\left(C_{n-1}^{n-1}n^n\right), \quad \text{for } t \in [0, T_0).$$

Thus we get for all $t < T_0$,

$$t \leq \left(2^{-1/n}\varepsilon^{(1-n)/n}nC_{n-1}^{(n-1)/n}\right)\beta(t)^{1/n} = \tfrac{1}{2}D_n\beta(t)^{1/n} \leq \tfrac{1}{2}D_n\beta(T_1)^{1/n},$$

so that

$$T_0 \leq \tfrac{1}{2}D_n\beta(T_1)^{1/n}.$$

Proof of the inequality (3.3). The inequality (3.3) with $n = 1$ and $C_1 = \frac{1}{2}$ follows from the cone inequality as $\operatorname{Diam} V \leq \frac{1}{2}\operatorname{Vol} V (= \frac{1}{2} \text{ length } V)$ for connected closed curves $V \subset \mathbf{R}^N$. (In fact, (3.3) is true with $n = 1$ and $C_1 = 1/(4\pi)$; see [17].)

Next by induction we assume the inequality (3.4) with n replaced by $n - 1$ to hold with some constant C_{n-1}. We further assume the n-dimensional manifold V in question to be *almost extremal*, that is, the ratio

$$\text{Fill Vol}(V \subset \mathbf{R}^N)/(\text{Vol } V)^{(n+1)/n}$$

to be as close to $C_{\max}$ as we wish. Take a hyperplane which divides V into two parts, say V_0 on "the left" of H and V_1 on "the right", and consider the intersection of V_1 with the levels of the function $d(x) = \text{dist}(x, H)$. Put

$$a(t) = V_1 \cap d^{-1}(t), \quad b(t) = V_0 + V_1 \cap d^{-1}[0, t],$$
$$\alpha(t) = \text{Vol } a(t), \quad \beta(t) = \text{Vol } b(t).$$

We span the cycles $a(t)$ by n-dimensional chains $\tilde{b}(t)$ of volumes

$$\tilde{\beta}(t) \leqslant C_{n-1}\alpha(t)^{n/(n-1)},$$

and then apply Lemma (A_3). As V is almost extremal, the weighted sum of the volumes

$$\left[\text{Vol}\left(b(t) - \tilde{b}(t)\right)\right]^{(n+1)/n} + \left[\text{Vol}\left(b'(t) + \tilde{b}(t)\right)\right]^{(n+1)/n}$$

for $b'(t) = V - b(t)$ can not be "much" smaller than $(\text{Vol } V)^{(n+1)/n}$, where "much" depends on how "almost extremal" is close to "extremal". Therefore either the volume $\beta(0) = \text{Vol } V_0$ is very small, or at least the half of the volume of V is contained between the hyperplanes H and $d^{-1}(T_0)$ on "the right" of H. As this applies to all hyperplanes parallel to H, we get almost all volume of V between a pair of such parallel hyperplanes with distance $\leqslant D_n(\text{Vol } V)^{1/n}$ between them.

As this conclusion holds for all families of hyperplanes in $\mathbf{R}^N$, almost the whole volume of V is contained in some ball $B(R)$ in $\mathbf{R}^N$ of radius

$$R \leqslant D_n(\text{Vol } V)^{1/n}.$$

Next we consider the intersection of V with concentric spheres $S(R + \rho)$, and using the coarea inequality we find a very small positive $\rho > 0$ such that the intersection

$$s(R + \rho) = V \cap S(R + \rho)$$

has "very small" $(n - 1)$-dimensional volume, that is, the ratio

$$(\text{Vol } s(R + \rho))^{n/(n-1)}/\text{Vol } V$$

is as close to zero as we wish (provided V is sufficiently close to "extremal"). We span the cycle $s(R + \rho)$ by the cone $\tilde{s}$ from the center of the ball $B(R)$, and thus divide V into the sum of two cycles:

$$V = V' + V'',$$

where V' is contained in the ball $B(R + \rho)$, and the volume of V' is artibrarily close to Vol V, while Vol V'' is negligibly small compared with vol V.

Finally, we span V' by the cone from the center of the ball, and span the "small" cycle V'' by a chain W'' of

$$\operatorname{Vol} W'' \leqslant C_N(\operatorname{Vol} V'')^{(n+1)/n},$$

according to the Federer-Fleming inequality. As V'' is small, the contribution of Vol W'' to the filling volume of V is nonessential, and thus

$$\operatorname{Fill\,Vol} V \leqslant \frac{R}{n+1}\operatorname{Vol} V \leqslant \frac{D_n}{n+1}(\operatorname{Vol} V)^{(n+1)/n}.$$

Hence we get (3.3) with

$$C_n = \frac{D_n}{n+1} \leqslant \frac{4n^2}{n+1} C_{n-1}^{(n-1)/n} < 4nC_{n-1}^{(n-1)/n},$$

and, by induction,

$$C_n < n^n.$$

3.4. Isoperimetric inequalities in Riemannian manifolds. The essential part of the argument of the previous section is a decomposition process of a cycle (or a submanifold) in $\mathbf{R}^N$ into a sum of "smaller" cycles. This process only depends on the coarea inequality, and so generalizes to n-dimensional cycles z in an arbitrary Riemannian manifold X.

We first consider all possible decompositions of a cycle z into a finite sum of n-dimensional *cycles*:

$$z = \sum_j z_j,$$

and then we try to minimize the sum

$$\Sigma = \sum_j (\operatorname{Vol} z_j)^{(n+1)/n},$$

over all such decomposition. We introduce the "weight" of z as the lower bound of the sums Σ over all decompositions of z:

$$\operatorname{Weight}(z) = \inf \Sigma.$$

Now we express the decomposition property of z in

3.4.A. Lemma. *Let every $(n-1)$-dimensional cycle in X have*

(3.5) $$\operatorname{Fill\,Vol} \leqslant C_{n-1}(\operatorname{Vol})^{n(n-1)}.$$

Then for every finite system of distance-decreasing functions d_i: $X \to \mathbf{R}$, $i = 1, \cdots, q$, (for example, for distance functions to some subsets $H_i \subset X$) and for every $\delta > 0$ there exists a decomposition of an arbitrary n-dimensional cycle z into *a finite sum of cycles*:

$$z = z_0 + \sum_{\mu=1}^{M} z_\mu,$$

such that the following three conditions are satisfied:

(a) $(\operatorname{Vol} z)^{(n+1)/n} \geqslant \operatorname{Weight}(z_0) + \Sigma_{\mu=1}^{M} (\operatorname{Vol} z_\mu)^{(n+1)/n}$,

(b) Weight $z_0 \leqslant \delta$,

(c) *every cycle* z_μ *is "sufficiently round" in the following sense. There is a decomposition of every* z_μ *into a sum of two chains*: $z_\mu = b'_\mu + b''_\mu$, *such that*

(i) $\operatorname{Vol}(b''_\mu) \leqslant \delta \operatorname{Vol} b'_\mu \leqslant \delta \operatorname{Vol} z_\mu$,

(ii) $(\operatorname{Vol} \partial b'_\mu)^{n/(n-1)} = (\operatorname{Vol} \partial b''_\mu)^{n/(n-1)} \leqslant \delta \operatorname{Vol} z_\mu$,

(iii) *the oscillation of every function* d_i, $i = 1, \cdots, q$, *on the support of* b'_μ *is at most*

$$D_n (\operatorname{Vol} b'_\mu)^{1/n} \quad \textit{for } D_n < 4n^2 C_{n-1}^{(n-1)/n}.$$

Proof. We start with a decomposition $z = \Sigma_j z_j$ for which the sum $\Sigma_j (\operatorname{Vol} z_j)^{(n+1)/n}$ is very close to the minimal value = Weight(z). Then take all those cycles z_μ among z_j, which satisfy the "roundness" condition (c), and denote by z_0 the sum of the remaining cycles z_j. The argument in (3.5) allows one to decompose "non-round" cycles into smaller cycles with substantial diminishing of the sum of $(\text{Volumes})^{(n+1)/n}$. Therefore the cycle z_0 has a negligibly small contribution to Weight z.

3.4.A′. Remark. If the manifold X is compact (possibly with boundary), then there is a finite system of distance functions d_i on X such that the "roundness" condition implies

$$\operatorname{Diam} b'_\mu \leqslant D_n (\operatorname{vol} z_\mu)^{1/n}.$$

3.4.B. Proposition. *Let a compact manifold X satisfy the inequality* (3.5). *Then every n-dimensional cycle z can be decomposed into a sum of cycles*:

$$z = z'_0 + \sum_{\mu=1}^{M} z'_\mu,$$

with the following three properties:

(a) $(\operatorname{Vol} z)^{(n+1)/n} \geqslant \operatorname{Weight} z'_0 + \Sigma_{\mu=1}^{M} (\operatorname{Vol} z'_\mu)^{(n+1)/n}$,

(b) Fill Vol $z'_0 \leqslant \delta$,

(c) Diam $z'_\mu \leqslant D_n (\operatorname{Vol} z'_\mu)^{1/n}$,

for $D_n < 4n^2 C_{n-1}^{(n-1)/n}$.

Proof. We start with the decomposition provided by Lemma 3.4.A. Then we span the boundaries $\partial b'_\mu$ by "small" chains $\tilde{c}_\mu$ according to the Sublemma below, thus further decomposing cycles z into the sums of cycles:

$$z_\mu = z'_\mu + z''_\mu,$$

for $z'_\mu = b'_\mu - \tilde{c}_\mu$ and $z''_\mu = b''_\mu + \tilde{c}_\mu$. The cycles z'_μ have almost the same volumes and diameters as z_μ, while the cycle $z'_0 = z_0 + \Sigma_\mu z''_\mu$ still has arbitrarily small "weight". Thus by the Sublemma the filling volume of z'_0 is also small.

3.4.B′. Sublemma. *For every compact manifold X there exists a small positive constant $\alpha = \alpha(X)$ such that every cycle y in X of volume less than α bounds a chain $\tilde{c}$ in X, which is "small" in the following sense*:

(i) $\operatorname{Vol} \tilde{c} \leqslant \tilde{C}(\operatorname{Vol} y)^{(k+1)/k}$,

for $k = \dim y$ and for some constant $\tilde{C} = \tilde{C}(X)$,

(ii) *the chain $\tilde{c}$ is contained in the ε-neighborhood of y for $\varepsilon \leqslant \tilde{C}(\operatorname{Vol} y)^{1/k}$.*

Proof. Take a C^2-smooth embedding of X into some space $\mathbf{R}^N$. By the Federer-Fleming theorem (see §3.4, Lemma 3.4.A) the cycle y bounds a "small" chain c in $\mathbf{R}^N$, which satisfies the inequalities (i) and (ii) with some constant $\overline{C}$ depending only on N and the embedding $X \to \mathbf{R}^N$. (If the imbedding is *path-isometric*, then $\overline{C}$ depends only on N.) If the volume of y is sufficiently small, then the chain c in $\mathbf{R}^N$ is close to $X \subset \mathbf{R}^N$, and so its normal projection to X satisfies (i) and (ii).

3.4.C. Corollary. *Let X be a compact manifold such that every k-dimensional cycle y in X, for $k = 1, \cdots, n$, satisfies the following "cone inequality"*:

$$\operatorname{Fill Vol} z \leqslant \frac{C}{k+1}(\operatorname{Diam} y)\operatorname{Vol} y,$$

for some constant $C = C(k) > 0$. Then every n-dimensional cycle z in X satisfies

$$\operatorname{Fill Vol} z \leqslant C_n(\operatorname{Vol} z)^{(n+1)/n}, \tag{3.6}$$

for $C_n < n^n \prod_{k=1}^n C(k)$.

Proof. Assume, by induction, the inequality (3.3) with n replaced by $n-1$, and then fill in the cycles z'_μ according to the cone inequality.

Additional remarks and corollaries.

(a) The cone condition with $C = 1$ is satisfied for compact convex subsets X in complete simply connected manifolds Y of *nonpositive sectional curvature*. Thus we obtain the inequality (3.6) with $C_n < n^n$ for these manifolds Y. This result is due to Hoffman-Spruk [45], who have originally proved this inequality by the method of Michael-Simon.

In fact the cone inequality with $C = 1$ (and thus the inequality (3.6) with $C_n < n^n$) also holds for complete simply connected manifolds X of nonpositive curvature, which have *next-to-convex boundary*, that is (see [33]), at most one of the principal curvatures of the boundary is negative. (The boundary is convex if and only if all principal curvatures are nonnegative.) For example, the boundary of every *surface* (dim $X = 2$) is next to convex, as well as the boundary of a Cartesian product of surfaces.

(b) The cone condition with $C(k) = k + 1$ holds for manifolds *without focal points*.

(c) The inequality (3.6) generalizes (see Appendix 2) to *complete* noncompact manifolds X, but the cone condition along without additional assumptions on X is not sufficient for (3.6), as simple examples show.

4. Filling in Banach spaces

We shall prove in this section an isoperimetric (filling) inequality for n-dimensional cycles in an arbitrary Banach space L. To state and prove such an inequality we need a notion of an n-dimensional volume of cycles in L. A particular choice of a volume may only affect the constant C_n in the isoperimetric inequality. However, it is convenient to fix some particular volume. A choice of the volume depends on a normalization of Haar's measures in the n-dimensional subspaces of L.

4.1. Normalization of Haar's measure. Let L_0 be an n-dimensional Banach space whose norm is denoted $\| \| = \| \|_{L_0}$. The Haar measure in L_0 is uniquely determined up to a positive multiple. In order to fix a Haar measure one must prescribe the total measure of some bounded measurable subset in L_0. For example, one defines the *Hausdorff measure* in L_0 by requiring the unit ball $B_{L_0} = \{\|X\|_{L_0} \leqslant 1,\ x \in L_0\}$ to have the same measure as the Euclidean unit ball in $\mathbf{R}^n$.

The measures mass and mass*. The choice of a measure in L_0 is equivalent to fixing a norm in the exterior power $\Lambda^n L_0$; the norm of an n-vector $x_1 \wedge \cdots \wedge x_n$ in $\Lambda^n L_0$ is interpreted as the total measure of the solid body spanned in L_0 by the vectors $x_1, \cdots, x_n$. The total measure (volume) of this body in the *Euclidean* space $\mathbf{R}^n$ satisfies Hadamard's inequality

$$\operatorname{Vol}(x_1 \wedge \cdots \wedge x_n) \leqslant \prod_{i=1}^{n} \|x_i\|_{\mathbf{R}^n}$$

with the equality for the frames of orthogonal vectors $x_1, \cdots, x_n$.

One *defines* the *mass norm* in the exterior power $\Lambda^n L_0$ as the upper bound of these norms $\| \|_\wedge$ on $\wedge^n L_0$, for which $\|x_1 \wedge \cdots \wedge x_n\|_\wedge \leqslant \prod_{i=1}^n \|X_i\|_{L_0}$. That is,

$$\operatorname{mass}(x_1 \wedge \cdots \wedge x_n) = \inf \prod_{i=1}^{n} \|x_i'\|_{L_0},$$

where the infimum is taken over the frames $(x'_1,\cdots,x'_n)$ which are obtained from the frame $(x_1,\cdots,x_n)$ by *unimodular* transformations of the space L_0.

The frames of independent vectors $x_1,\cdots,x_n$ in L_0 for which $\operatorname{mass}(x_1\wedge\cdots\wedge x_n)=\prod_{i=1}^n\|x_i\|$ are said to be (*mass*) *extremal*. If $(e_1,\cdots,e_n)$ is an extremal frame of *unit* vectors ($\|e_i\|=1$), then one introduces the associated (*mass*) extremal l^∞-norm in L_0:

$$\left\|\sum_{i=1}^n a_ie_i\right\|_{l^\infty}=\max_{1\leqslant i\leqslant n}|a|_i.$$

An elementary argument shows that $\|x\|_{l^\infty}\leqslant\|x\|_{L_0}$ for all vectors $x\in L_0$. That is, the unit ball B_{L_0} is contained in the l^∞-ball

$$B_{l^\infty}=\left\{x=\sum_{i=1}^n a_ie_i\in L_0\,\middle|\,|a_i|\leqslant 1,\ i=1,\cdots,n\right\}.$$

We also consider the *extremal l^1-norm* in L_0,

$$\left\|\sum_{i=1}^n a_ie_i\right\|_{l^1}=\sum_{i=1}^n|a_i|,$$

and we have $B_{l^1}\subset B_{L_0}\subset B_{l^\infty}$. Hence

$$2^n/n!=\operatorname{mass}_{L_0}B_{l^1}\leqslant\operatorname{mass}_{L_0}B_{L_0}\leqslant\operatorname{mass}_{L_0}B_{l^\infty}=2^n,$$

for an arbitrary n-dimensional space L_0. Furthermore, $\operatorname{mass}_{L_0}=\operatorname{mass}_{l^1}$ and so the space l^1 has the unit ball of the minimal possible mass $=2^n/n!$. However, the frame $e_1,\cdots,e_n$, which is extremal for $\operatorname{mass}_{L_0}$, is not extremal ($n\geqslant 2$) for $\operatorname{mass}_{l^\infty}$. In fact, for n even, one can take for a $\operatorname{mass}_{l^\infty}$-extremal frame a system of n great diagonals $d_j=\sum_{i=1}^n\pm e_i$, $j=1,\cdots,n$, of the cube (ball) B_{l^∞}, such that these vectors (diagonals) d_j are mutually orthogonal in *the l^2-metric.* Thus

$$\left\|\sum_{i=1}^n a_ie_i\right\|_{l^2}\stackrel{\text{def}}{=}\sqrt{\textstyle\sum_{i=1}^n a_i^2},$$

and therefore

$$\operatorname{mass}_{l^\infty}B_{l^\infty}=n^{-k}2^n\quad\text{for } n=2k=\dim l^\infty.$$

Now for a smooth n-dimensional submanifold V in any Banach space one has, with the masses in the tangent spaces $L_v=T_v(V)\subset L$, $v\in V$, a fixed measure, called *mass* on V. Moreover, one has this mass for all piecewise smooth submanifolds in L as well as for singular piecewise smooth chains in L with real, integral or with $\mathbf{Z}_2$-coefficients.

This mass obviously satisfies the following inequality for the linear cone over V from any point $x \in L$:

$$\operatorname{mass}(\text{cone}) \leqslant \frac{1}{n+1} \operatorname{dist}(x, V) \operatorname{mass} V.$$

Moreover,

$$\operatorname{mass}(\text{cone}) \leqslant \frac{1}{n+1} \int_{n+1} \int_V \operatorname{dist}(x, v)\, dv,$$

for $dv = d(\text{mass})$ on V.

Next we define another measure in L_0, called *mass**, as the dual of the mass in the space L_0^* dual to L_0. Recall that the dual space L_0^* consists of the linear functionals $x^*: L_0 \to \mathbf{R}$, and the norm $\|x^*\|_{L_0^*}$ for $x^* \in L_0^*$ is

$$\|x^*\|_{L_0^*} \stackrel{\text{def}}{=} \sup_{\|x\| \leqslant 1} |x^*(x)|.$$

Then we consider *dual* frames of independent vectors, $x_1, \cdots, x_n$ in L_0 and $x_1^*, \cdots, x_n^*$ in L_0^*, for which $x_i^*(x_j) = \delta_{ij}$, and we put

$$\operatorname{mass}^*(x_1 \wedge \cdots \wedge x_n) \stackrel{\text{def}}{=} \left[\operatorname{mass}_{L_0^*}(x_1^* \wedge \cdots \wedge x_n^*)\right]^{-1}.$$

If $e_1^*, \cdots, e_n^*$ is a mass-extremal frame of unit vectors in L_0, then the dual vectors $e_1, \cdots, e_n$ in L_0 also have unit norm so that $\|e_i\|_{L_0} = 1$. These dual frames $(e_1, \cdots, e_n)$ in L_0 are called *mass*-extremal*. The l^∞-norms in L_0 associated to these frames are called *mass*-extremal l^∞-norms*. These norms can be geometrically described as follows. Consider all l^∞-norms in L_0 which are no greater than $\| \ \|_{L_0}$. As every l^∞-norm is determined by its ball, which is a solid body isomorphic to a Euclidean cube, one could equally consider all such bodies in L_0, which contain the unit ball B_{L_0}. Fix any Haar measure (volume) in L_0 and take the l^∞-norms corresponding to the bodies of *minimal* volume. These are exactly the mass*-extremal l^∞-norms, as it follows from the mass-mass* duality. In particular, the mass*-extremal l^∞-norms satisfy

$$\| \ \|_{l^\infty} \leqslant \| \ \|_{L_0}.$$

Furthermore, if the space L_0 is an l^∞-space to start with, then the extremal l^∞-norm equals the original norm $\| \ \|_{L_0}$. It follows, now for an arbitrary space L_0, that every mass*-extremal l^∞-norm satisfies

$$\operatorname{mass}^*_{L_0}(B_{L_0}) \leqslant \operatorname{mass}^*_{L_0}(B_{l^\infty}) = \operatorname{mass}^*_{l^\infty}(B_{l^\infty}) = 2^n,$$

and that the measures mass and mass* are related by the following inequalities

$$\operatorname{mass}_{L_0} \leqslant \operatorname{mass}^*_{L_0} \leqslant n^{n/2} \operatorname{mass}_{L_0}.$$

Moreover, if L_0 is an even dimensional l^∞-space, then mass* $= n^{n/2}$ mass. For example, the two-dimensional l^∞-space, whose unit ball B_{l^∞} is the Euclidean square, has mass*$B_{l^\infty} = 2$ mass $B_{l^\infty} = 4$. On the other hand, the two-dimensional space whose unit ball is the regular $2k$-gon for k *odd*, has mass $=$ mass*. The most interesting space is the "6-gonal" space (plane), as the corresponding unit ball (6-gon) has mass* $= 3$. Any other 2-dimensional space L_0 has mass*$B_{L_0} \geqslant 3$, (see below).

The most important properties of the mass* are the *compressing property* and the *coarea inequality*.

Compressing. Take an n-dimensional subspace L_0 in a Banach space L. There exists, by the Hahn-Banach theorem, a linear projection $P\colon L \to L_0$ which is distance-decreasing relative to the mass*-extremal l^∞-norm in L_0. Therefore this P also is *mass*-decreasing* (compressing) on all n-dimensional subspaces in L. (This property is closely related to *Almgren's ellipticity condition* [3].)

The compressing property yields the following generalization of Proposition 2.1.A.

Let X be an arbitrary submanifold in a Banach space, Y a subset in X, and f a map of Y to some n-dimensional Banach space L_0 such that f is distance-decreasing relative to some mass*-extremal l^∞-norm in L_0. (If f is distance-decreasing relative to the norm $\| \; \|_{L_0}$, then it is also distance-decreasing for every extremal l^∞-norm in L_0.) *Then the map f extends to a Lipschitz map $X \to L_0$ which is mass*-decreasing on all n-dimensional submanifolds of X.*

Proof. We realize L_0 as a subspace in some L^∞-space. Then the map f extends to the distance-decreasing map $X \to L^\infty$, and the compressing projection $L^\infty \to L_0$ applies.

Coarea inequality. Consider the solid $x_1 \wedge \cdots \wedge x_n$ spanned by n independent vectors $x_1, \cdots, x_n$ in some Banach space L. Let h_1 denote the first height of this solid, which is the distance from x_1 to the $(n-1)$-dimensional space spanned by the vectors $x_2, \cdots, x_n$.

The linear coarea inequality claims

$$\text{mass}^*(x_1 \wedge \cdots \wedge x_n) \geqslant h_1 \, \text{mass}^*(x_2 \wedge \cdots \wedge x_n),$$

that is, the n-dimensional mass* of the solid is not less than the product of the $(n-1)$-dimensional mass* of the base of the height of the solid.

Proof. We may assume the space L to be equal to the span of the vectors $x_1, \cdots, x_n$. Then the heights $h_1, \cdots, h_n$ equal the norm of the dual vectors, i.e.,

$$h_i = \|x_i^*\|_{L^*},$$

and by the definition of mass*

$$\text{mass}^*(x_1 \wedge \cdots \wedge x_n) \geqslant \prod_{i=1}^{n} h_i.$$

We may further assume that the vectors $x_2, \cdots, x_n$ form a mass*-extremal basis in the $(n-1)$-dimensional subspace L' spanned by these vectors. Then we take a vector x_1' in the (one-dimensional) kernel of the above mass*-compressing projection $L \to L'$ such that

$$\text{mass}^*(x_1' \wedge x_2 \wedge \cdots \wedge x_n) = \text{mass}(x_1 \wedge x_2 \wedge \cdots x_n).$$

The new solid $x_1' \wedge x_2 \wedge \cdots \wedge x_n$ has the same first height $h_i' = h_1$, while the heights h_i' for $i \geqslant 2$ are equal to the $n-1$ heights of the solid $x_2 \wedge \cdots \wedge x_n$, because the projection $L \to L'$ is distance-decreasing relative to the extremal l^∞-metric in L'. As the basis $x_2, \cdots, x_n$ is mass*-extremal, the products of the heights $h_2', \cdots, h_n'$ of the solid $x_2 \wedge \cdots \wedge x_n$ equals the mass* of this solid. Therefore

$$\begin{aligned} \text{mass}^*(x_1 \wedge \cdots \wedge x_n) &= \text{mass}^*(x_1' \wedge x_2 \wedge \cdots \wedge x_n) \\ &\geqslant \prod_{i=1}^{n} h_i' = h_1 \prod_{i=2}^{n} h_i' = h_1 \, \text{mass}^*(x_2 \wedge \cdots \wedge x_n). \end{aligned}$$

The "linear" inequality yields the following coarea inequality for the $(n-1)$-dimensional levels of each distance-decreasing function $d\colon V \to \mathbf{R}$ for an n-dimensional submanifold V in some Banach space L:

$$\text{mass}^* V \geqslant \int_{-\infty}^{+\infty} \text{mass}^* d^{-1}(t)\, dt.$$

This inequality applies, for example, to the distance function $d(v) = \text{dist}(v, H)$ to a fixed subset $H \in L$.

Remark. The definitions of mass and mass* apply to an arbitrary *Finsler* manifold V, as it can be isometrically imbedded into the Banach space $L^\infty(V)$ (see §1.1). The above properties of mass and mass* hold for all Finsler manifolds.

Examples. The mass* of the unit ball is an n-dimensional Banach space L_0 is related to the $(n-1)$-dimensional mass of the boundary by the inequality

$$\text{mass}^* B_{L_0} \geqslant \frac{1}{n+1} \text{mass}^* \partial B_{L_0},$$

which is opposite to the obvious inequality for the mass:

$$\text{mass}\, B_{L_0} \leqslant \frac{1}{n+1} \text{mass}\, \partial B_{L_0}.$$

In particular, if $n = 2$, then

$$\text{mass}^* B_{L_0} \geqslant \tfrac{1}{2} \text{ length } \partial B_{L_0},$$

(while mass $B_{l_0} \leqslant \frac{1}{2}$ length ∂B_{L_0}). As any such boundary ∂B_{l_0} has length $\geqslant 6$ (see [67]), we get the relation mass*$B_{L_0} \geqslant 3$ for every two-dimensional Banach space L_0.

Remark. The coarea inequality generalizes (by induction) to systems of functions

$$F = (f_1, \cdots, f_k) \colon V \to \mathbf{R}^k,$$

where the receiving space $\mathbf{R}^k$ is equipped with the l^∞-norm for which

$$\text{Dilation } F = \max_{1 \leqslant i \leqslant k} \text{Dilation}(f_i).$$

Namely, if F is a distance-decreasing map (Dil $\leqslant 1$), then

$$\text{mass}^* V \geqslant \int_{\mathbf{R}^k} \text{mass}^*\big(F^{-1}(x)\big)\, dx.$$

The (hyper-Euclidean) volume. There are several canonical ways to equip an n-dimensional Banach space L_0 with a Euclidean norm. The most "popular" norm comes from the canonical embedding of L_0 into the Hilbert space of functions on the dual ball $B_L \subset L_0^*$. The vectors of L_0 go to linear functions (functionals) on the ball $B_{L_0^*}$.

We shall use another standard Euclidean norm on L_0, namely, the l^2-norm $\|\ \|_{l^2} \geqslant \|\ \|_{L_0}$ which maximizes the volume of the l^2-ball $B_{l^2} \subset B_{L_0}$ relative to some fixed Haar measure in L_0. The Euclidean volume associated to this l^2-norm is called the (hyper-Euclidean) volume in L_0. This volume clearly satisfies

$$\text{mass}^* \leqslant \text{Vol} \leqslant n^{n/2}\, \text{mass}.$$

The equality mass* $=$ Vol holds for $L_0 = \mathbf{R}^n$ and also for $L_0 = l^\infty$. The equality Vol $= n^{n/2}$ mass holds only for the n-dimensional l^1-space.

The unit ball B_{L_0} always has volume $\geqslant$ the volume of the unit Euclidean ball, where the equality is possible only for $L_0 = \mathbf{R}^n$.

This volume for submanifolds $V \subset L$ agrees with the one given in §2. Furthermore, this volume enjoys the same coarea formula as the mass* and also the following "cone" inequality (compare §3.3) for closed n-dimensional submanifolds V in (finite or infinite dimensional) L^∞-spaces:

$$\text{Fill Vol}(V \subset L^\infty) \leqslant \text{const}_n (\text{Vol } V) \text{Diam}(V).$$

for $\text{const}_n = \frac{1}{\pi} \int_0^{\pi/2} (\cos x)^n\, dx < 1/\sqrt{n+1}$.

Proof. Suppose for the moment that V is a submanifold in the boundary of a hemisphere of constant curvature:

$$V \subset S^N = \partial S_+^{N+1} \subset S_+^{N+1}.$$

Then the geodesic cone C_+ from the North pole $p_+ \in S_+^{N+1}$ over V has

$$\operatorname{Vol} C_+ = \operatorname{const}_n D \operatorname{Vol} V,$$

for $D = \operatorname{Diam} S^N$. As the embedding $V \hookrightarrow L^\infty$ extends to a distance-decreasing map $C_+ \to L^\infty$ we get a "cone" in L^∞ over V of volume $\leqslant \operatorname{const}_n D \operatorname{Vol} V$.

Now the distance between a pair of points in the cone C_+, $c_1 = (v_1, t_1)$ and $c_2 = (v_2, t_2)$, for $v_1, v_2 \in V$ and t_1 and t_2 in the iterval $[0, \frac{1}{2}D]$, only depends on t_1, t_2 and $\operatorname{dist}(v_1, v_2)$. Therefore for *any* metric in V we can assign this distance to the pairs of points c_1 and c_2 of the *abstract* cone as long as $\operatorname{dist}(v_1, v_2) \leqslant D$. The abstract cone with this distance goes to $L^\infty \supset V$ as above, and the proof is concluded.

4.2. Isoperimetric inequalities in Banach spaces. We prove in this section an inequality for the filling mass* in an arbitrary Banach space L and also a somewhat sharper inequality for the filling volume in L^∞-spaces.

4.2.A. Theorem. *An arbitrary piecewise smooth n-dimensional cycle z in a Banach space L satisfies*

$$\operatorname{Fill\,mass}^* z \leqslant C_n (\operatorname{mass}^* z)^{(n+1)/n},$$

for $C_n < n^n \prod_{k=1n} (k+1)^{(k+1)/2}$.

Proof. First we observe the following cone inequality for k-dimensional cycles z_k:

$$\operatorname{Fill\,mass} z_k \leqslant (k+1)^{(k+1)/2-1} (\operatorname{Diam} z_k) \operatorname{mass}^* z_k.$$

Indeed, the mass of the cone over z_k from some point in the support of z_k is at most

$$\frac{1}{k+1} (\operatorname{Diam} z_k) \operatorname{mass} z_k \leqslant \frac{1}{k+1} \operatorname{Diam} z_k (\operatorname{mass}^* z_k),$$

while $\operatorname{mass}_{k+1}^* \leqslant (k+1)^{(k+1)/2} \operatorname{mass}_{k+1}$ (see §4.1).

Next every piecewise smooth cycle z can be approximated by piecewise linear cycles. Every piecewise linear cycle is contained in a finite dimensional subspace of L, and we may assume that the space L is finite dimensional to start with. Then L is isomorphic to the Euclidean space $\mathbf{R}^N$ for $N = \dim L$, and so the cycles z in L satisfy the Federer-Fleming inequality

$$\operatorname{Fill\,mass}^* z \leqslant \bar{C} (\operatorname{mass}^* z)^{(n+1)/n},$$

for some constant $\bar{C} = \bar{C}(L)$. (In fact, if one uses the isomorphism $L \xrightarrow{\sim} \mathbf{R}^N$ provided by the mass-extremal frame in L, then one gets $\bar{C} = \bar{C}(N)$.)

Finally, we invoke the coarea inequality for mass* (see §4.1), and then observe that our proof of (3.3) (also compare §3.4) extends to space L with the additional factor $(k+1)^{(k+1)/2}$ in every dimension $k = 1,\cdots,n$, which is due to the new cone inequality.

4.2.B. *If the space L is isometric to an (finite or infinite dimensional) L^∞-space, then*

$$\text{Fill Vol } z \leqslant C_n(\text{Vol } z)^{(n+1)/n}$$

for

$$C_n < n!\sqrt{(n+1)!}\,.$$

Indeed, the above proof applies with an obvious modification due to the cone inequality for the volume (see §4.1).

As a corollary we obtain the filling volume inequality of §2.3 for Riemannian manifolds V. Observe that the canonical embedding $V \subset L^\infty(V)$ admits the following simple approximation by imbeddings into finite dimensional subspaces of L. One takes an ε-net of N points $v_1,\cdots,v_N$ in V, for a small positive $\varepsilon > 0$ and a large N, and one maps V to the N-dimensional space $l^\infty\{v_1,\cdots,v_N\}$ by sending

$$v \to (\text{dist}(v, v_1),\cdots,\text{dist}(v, v_N)),$$

for all $v \in V$.

4.3. Filling radius in Banach spaces. First let L be a finite dimensional Banach space. Then using an isomorphism $L \xrightarrow{\sim} \mathbf{R}^N$ and Remark 3.2.A we obtain

$$\text{Fill Rad } z \leqslant \text{const}_L(\text{Vol } z)^{1/n},$$

for an arbitrary n-dimensional cycle z in L.

Next the argument of §2.4 applies to the distance function $d(x) =$ dist(x, support z), or rather to the levels $c \cap d^{-1}(t)$ of the "almost minimal" $(n+1)$-dimensional chain c which spans z, and now that argument does yield the inequality

$$\text{Fill Rad } z \leqslant (n+1)C_n(\text{Vol } z)^{1/n},$$

where $C_n = C_n(L)$ denotes the upper bound of the ratio

$$\text{Fill Vol } z/(\text{Vol } z)^{(n+1)/n},$$

over all n-dimensional cycles z in L. In fact, the analytic Lemma (A_1) of §3.3. implies the existence of a value $t_0 \leqslant (n+1)C_n(\text{Vol } z)^{1/n}$, for which the volume of $1/n$ the cycle $c \cap d^{-1}(t_0)$ becomes arbitrary (depending on "almost minimality" of c) small, and this "residual" small cycle is spanned according to Remark 3.2.A. Thus we obtain a chain c', which spans z such that the following hold:

(i) $\operatorname{Vol} c' \leqslant \operatorname{Fill} \operatorname{Vol} z + \delta$,

for an arbitrary small $\delta > 0$,

(ii) the support of c is contained in the ε-neighborhood U_ε (support z) for

$$\varepsilon - \delta \leqslant (n+1)C_n^{n/(n+1)}(\operatorname{Fill} \operatorname{Vol} z)^{1/(n+1)} \leqslant (n+1)C_n(\operatorname{Vol} z)^{1/n}.$$

This argument also applies to mass* (as mass* satisfies the coarea inequality), and yields the following theorems for finite and infinite dimensional Banach spaces.

4.3.A. *An arbitrary n-dimensional cycle z in a Banach cycle L has*

$$\operatorname{Fill} \operatorname{Rad}(z \subset L) \leqslant \operatorname{const}_n (\text{mass}^* z)^{1/n},$$

for $\operatorname{const}_n < (n+1)n^n \prod_{k=1}^n (k+1)^{(k+1)/2}$.

4.3.B. *If the space L is isometric to an* L^∞*-space, then*

$$\operatorname{Fill} \operatorname{Rad}(z \subset L) \leqslant \operatorname{const}_n (\operatorname{Vol} z)^{1/n},$$

for $\operatorname{const}_n < (n+1)n^n\sqrt{(n+1)!}$.

We obtain as a corollary our main estimate (Main theorem 1.2.A) for the filling radius of a Riemannian manifold V, and hence the upper bound 0.1.A for the shortest geodesic in an essential manifold. We get, moreover, the following relation:

$$\operatorname{Fill} \operatorname{Rad} V \leqslant \operatorname{const}'_n (\operatorname{Fill} \operatorname{Vol} V)^{1/(n+1)}, \tag{4.1}$$

for $n = \dim V$ and

$$\operatorname{const}'_n < (n+1)\left(n^n\sqrt{(n+1)\,!}\right)^{n/(n+1)}.$$

4.3.C. Minimal fillings in Banach spaces. Let V be a closed n-dimensional submanifold (or sub-pseudomanifold) in a Banach space L, and let c_i, $i = 1, 2, \cdots$, be a minimizing sequence of chains which span V, that is, $\partial c_i = V$ and $\operatorname{Vol} c_i \to \operatorname{Fill} \operatorname{Vol}(V \subset L)$ for $i \to \infty$. Such a sequence is said to *converge*, if there exists a *Hausdorff limit* (see Appendix 3) of the supports, $\operatorname{supp} c_i \subset L$, which is a compact subset W in L such that Hausdist $(W, \operatorname{supp} c_i) \to 0$ for $i \to \infty$. Such a limit W of a minimizing sequence c_i is called a *minimal filling*, if no proper subset of W is a limit of any other minimizing sequence c'_i in L.

4.3.C′. Lemma. *Every (pseudo-) manifold in a finite dimensional space L admits a minimal filling.*

Proof. Take a minimizing sequence of chains c_i which have supports in a fixed bounded neighborhood of V. Then there is a subsequence whose supports converge to a subset W_0 in L. Take for W a minimal compact subset in W_0, which is the limit of some minimizing sequence.

Let W be a minimal filling of V which is the limit of a minimizing sequence c_i. We may assume (by passing to a subsequence if necessary) that the volume-measures on (the supports of) c_i *weakly converge* to a measure, called the *volume*, on W. That is, the volume of every open subset W_0 in W is the limit of the intersections of the ε-neighborhoods of W_0 with c_i. Thus

$$\operatorname{Vol} W_0 = \lim_{\substack{\varepsilon \to 0 \\ i \to \infty}} \operatorname{Vol}[U_\varepsilon(W_0) \cap c_i].$$

Using this definition we have

$$\operatorname{Vol} W = \operatorname{Fill Vol}(V \subset L),$$

and so W may be viewed as a "minimal variety" which spans V.

4.3.C″. Theorem. *The intersection of a minimal filling with a ball has*

$$\operatorname{Vol}(B_w(R) \cap W) \geqslant A_n R^{n+1} \tag{4.2}$$

for all $w \in W$, $R \leqslant \operatorname{dist}(w, V)$, *and* $A_n = (n+1)^{-n+1} C_n^{-n}$, *where* C_n *is the constant of* 4.2.*B*.

Proof. We must establish the corresponding inequality for approximating chains c_i at some points $w_i \to w$ in $\operatorname{supp} c_i$.

Let $c_i(R) = c_i \cap B_{w_i}(R)$ and $z_i(R) = \partial c_i(R) = c_i \cap \partial B_{w_i}(R)$. As the sequence c_i is minimizing, we have for $i \to \infty$

$$[\operatorname{Vol} c_i(R) - \operatorname{Fill Vol} z_i(R)] \to 0,$$

for every fixed $R < \operatorname{dist}(w, V)$. As $\operatorname{Fill Vol} z_i(R) \leqslant C_n[\operatorname{Vol} z_i(R)]^{(n+1)/n}$, using the coarea inequality and Lemma (A_1) of §3.3 we conclude that the inequality $\operatorname{Vol} c_i(R) \leqslant A'_n R_n$, for a fixed $A'_n < A_n$, is possible only if $\operatorname{Vol}(c_i(R)) \to 0$ for $i \to \infty$. Then, for some R' close to R, the cycles $z_i(R')$ also have volumes $\to 0$, and so they can be filled by chains c'_i, which have

$$\operatorname{Vol} c'_i \leqslant \operatorname{Vol} c_i(R'),$$

and are supported in ε_i-neighborhoods of $z_i(R)$ for $\varepsilon_i \to 0$ as $i \to \infty$. Then the limit of the supports of the sequence $c_i - c_i(R') + c'_i$ is a proper subset of W. This contradicts the minimality of W, and so the proof is concluded.

4.3.C″. Corollary. *Every (pseudo-) manifold V in an arbitrary space L^∞ admits a minimal filling W which satisfies the above inequality* (4.2).

Proof. Take some approximations V_j to V, which are contained in finite dimensional subspaces of L^∞ and take some minimal fillings W_j of V_j. The inequality (4.2) implies the *uniform compactness* of the sequence W_j (see [37] and [34]), and so some subsequence of W_j admits *an abstract Hausdorff limit* (See Appendix 3) W. This W can be sent into the space L^∞ by a distance-decreasing map (see §1.1), which extends the embedding $V \hookrightarrow L^\infty$.

Remark. The minimal filling W of V does not solve the Plateau problem, as the "natural" volume of W may be greater than the weak limit of the volumes of the minimizing chains. In fact, this W is just a formal device, which summarizes some important properties of minimizing sequences.

On the other hand, if one uses mass* in place of the volume, then one can construct an actual mass*-minimizing filling of V by an additional regularization of the above procedure.

4.3.C. Let us indicate an application of the inequality (4.1). Recall that (see §4.1)

$$\mathrm{Vol}_{n+1} \leqslant (n+1)^{(n+1)/2}\mathrm{mass}_{n+1}.$$

Since the mass satisfies the cone inequality, we obtain the following relation for the volumes of the cones over $V \subset L^\infty(V)$ from the points $v \in V$:

$$\mathrm{Vol}(\mathrm{cone}_v) \leqslant (n+1)^{(n+1)/2-1}\int_V \mathrm{dist}(v, w)\, dw.$$

Therefore

$$\mathrm{Fill\,Vol}\, V \leqslant (n+1)^{(n-1)/2}(\mathrm{Vol}\, V)^{-1}\iint_{V\times V} \mathrm{dist}(v, w)\, dv\, dw,$$

and as

$$\mathrm{Fill\,Vol} \leqslant n^n\sqrt{(n+1)\,!}\,(\mathrm{Vol})^{(n+1)/n},$$

we get

$$\mathrm{Fill\,Vol} \leqslant B_n\left[\iint \mathrm{dist}(v, w)\, dv\, dw\right]^{(n+1)/(2n+1)}.$$

for

$$B_n < (n+1)^{(n^2-1)/(4n+2)}n^{n^2/(2n+1)}\left(\sqrt{(n+1)\,!}\right)^{n/(2n+1)} < (n+1)^{n+1}.$$

Hence

$$\mathrm{Fill\,Rad}\, V \leqslant B_n'\left[\iint \mathrm{dist}(v, w)\, dv,\, dw\right]^{1/(2n+1)},$$

for $B_n' < (n+1)^2(n^n\sqrt{(n+1)\,!})^{n/(n+1)}$.

Corollary (*Compare* §1.2). *The length of the shortest noncontractible geodesic γ in an n-dimensional essential manifold V satisfies*

$$\mathrm{length}\,\gamma \leqslant 6B_n'\left[\iint_{V\times V} \mathrm{dist}(v, w)\, dv\, dw\right]^{1/(2n+1)},$$

for the above constant B_n'.

4.4. Isosystolic inequalities for open manifolds. Let V be a (possibly noncompact) manifold with boundary. We say that V is *essential relative to infinity* if there is a continuous map f of V to some aspherical space K, such that f is constant outside some compact subset in the interior of V and the map $f: V \to K$ represents a nonzero n-dimensional homology class in K for $n = \dim V$.

Examples. (a) The complement $V = W \setminus C$ to a closed simply connected subset C in a closed essential manifold W is essential relative to infinity.

(b) If a manifold V contains an open subset which is essential relative to infinity, then V itself is essential relative to infinity.

4.4.A. Theorem. *Let V be a complete (as a metric space) Riemannian manifold. If V is essential relative to infinity, then there exists a closed noncontractible curve γ in V such that*

$$\text{length}\,\gamma \leqslant A\left[\text{Vol}\,V + B(\text{Vol}\,\partial V)^{n/(n-1)}\right]^{1/n},$$

for $n = \dim V$ and some constants

$$A < 6(n+1)n^n\sqrt{(n+1)!}\,, \quad B < (n-1)^{n-1}\sqrt{n!}\,.$$

Furthermore, if the boundary ∂V is empty, then there is a closed geodesic γ in V of

$$\text{Length}\,\gamma \leqslant A(\text{Vol}\,V)^{1/n}.$$

Proof. Take concentric spheres $S(R)$ around a fixed point $v \in V$. As

$$\int_0^\infty S(R)\,dR = \text{Vol}\,V < \infty,$$

there are arbitrary large spheres $S(R)$ of arbitrary small $(n-1)$-dimensional volume. We smooth such a sphere $S(R)$ and replace V by the compact region inside the smoothed sphere. Thus we reduce the problem to the case of a compact manifold V with boundary.

Next we fill in the boundary ∂V of V by an n-dimensional chain, which is interpreted as a pseudo-manifold V' with boundary $\partial V' = \partial V$ such that

$$\text{Vol}\,V' \leqslant B(\text{Vol}\,\partial V)^{n/(n-1)}.$$

Thus we get a closed pseudomanifold $W = V \cup V'$ which has

$$\text{Fill Rad}\,W \leqslant 6A(\text{Vol}\,W)^{1/n},$$

for $\text{Vol}\,W = \text{Vol}\,V + \text{Vol}\,V'$.

This manifold W admits an essential map $f: W \to K$ which is constant on V', and the argument of Lemma 1.2.B provides a curve γ' in W of length $\leqslant 6$ Fill Rad W, whose image under f is not contractible in K. Therefore one of the segments of γ' inside V with the ends on the boundary ∂V gives a noncontractible loop in K. As the filling V' of ∂V respects the metric in $\partial V'$, this segment can be completed to a closed noncontractible curve γ in V of length $\gamma \leqslant$ length γ'.

Finally, if the original manifold V has no boundary, then the above curve γ can not be homotoped to infinity, and so there is a closed geodesic in V of length $<$ length γ which is homotopic to γ.

Open manifolds with essential ends. A closed connected submanifold H in V is said to be *essential in V* if there exists a map of H into an aspherical space $f\colon H \to K$ such that the kernel of the homomorphism $f_*\colon \pi_1(H) \to \pi_1(K)$ contains the kernel of the inclusion homomorphism $\pi_1(H) \to \pi_1(V)$, and such that the image of the fundamental homology class $f_*[H] \subset H_*(K)$ is nonzero. For example, if H is an essential manifold and the fundamental group $\pi_1(H)$ injects into $\pi_1(V)$, then H is essential in V.

Let V be a connected open manifold without boundary and let $d\colon V \to \mathbf{R}_+$ be a proper Morse function. We say that *V has at least k essential ends* if, for some regular value $t \in \mathbf{R}_+$, there are k disjoint connected noncompact components in the set $d^{-1}[t, \infty) \subset V$, say $\overline{H}_i$, $i = 1, \cdots, k$, such that every $\overline{H}_i$ has a boundary component, say $H_i \subset \partial \overline{H}_i \subset d^{-1}(t) \subset V$, which is an essential submanifold (of codimension one) in V.

This definition does not depend on a particular Morse function d on V. Furthermore, one can choose, if one wishes, the above value t arbitrary large.

Examples. (a) If V_0 is a closed essential manifold, then the product $V_0 \times \mathbf{R}$ has two essential ends. The connected sum of k-copies of $V_0 \times \mathbf{R}$ has $2k$ essential ends.

(b) If V admits a complete metric of constant negative curvature and finite total volume, then every end of V is essential. That is, the number of essential ends equals the number of cusps of V. The connected sum of infinitely many of such manifolds V has infinitely many essential ends.

4.4.B. *Let V be an open complete Riemannian manifold with at least k essential ends. Consider concentric balls $B(R)$ in V around a fixed point $v_0 \in V$ and let*

$$\liminf R^{-1} \operatorname{Vol} B(R) = M < \infty.$$

Then the first systole $\operatorname{sys}_1 V$, *that is, the lower bound of the lengths of noncontractible curves in V, satisfies*

$$\operatorname{sys}_1 V \leqslant 6(n-1) C_{n-1} (M/k)^{1/(n-1)},$$

for $n = \dim V$ and $C_{n-1} = (n-1)^{n-1}\sqrt{n!}$.

Proof. Let $d = d(v)$ be a smooth approximation to the distance function $\operatorname{dist}(v, v_0)$ to a fixed point $v_0 \in V$. Then there are arbitrary large values t for which $\operatorname{Vol} d^{-1}(t) \leqslant M = \varepsilon$ for an arbitrary small $\varepsilon > 0$. We have an essential component H_t in $f^{-1}(t)$ of volume $\leqslant (M/k) + \varepsilon$, and we obtain by the argument of 1.3 a required short curve in H_t, which is not contractible in V.

4.4.C. Filling radius of complete noncompact manifolds. If V is a complete noncompact manifold, we denote by $L^\infty(V)$ the space of *all* (not only bounded) Borel functions on V, and then we have our canonical imbedding $V \subset L^\infty(V)$, that is,

$$v \to f_v(w) = \operatorname{dist}(v, w).$$

Observe that this space $L^\infty(V)$ is not, strictly speaking, a metric space, as the distance between two functions f_1 and f_2 on V,

$$\operatorname{dist}(f_1, f_2) \stackrel{\text{def}}{=} \sup_{v \in V} |f_1(v) - f_2(v)|,$$

may be infinite. However, our embedding $V \subset L^\infty(V)$ is isometric just the same.

In order to define the filling radius of V in $L^\infty(V)$, we allow only those infinite chains c which are locally finite in $L^\infty(V)$, i.e., every bounded subset in $L^\infty(V)$ intersects only finitely many (supports of) singular simplices of c. Then we define the filling radius of $V \subset L^\infty(V)$ as the lower bound of those $\varepsilon > 0$, for which there exists a locally finite $(n+1)$-dimensional (for $n = \dim V$) chain c in the ε-neighborhood $U_\varepsilon(V) \subset L^\infty(V)$, whose boundary ∂c is contained in V and which represents the fundamental class of V in the homology group $H_n(V)$ with *noncompact supports*. Geometrically speaking, the condition $\operatorname{Fill\,Rad} V \leqslant \varepsilon$ is equivalent to the existence of a pseudomanifold W with boundary $\partial W = V$ such that the metric of W extends that of V and such that $\operatorname{dist}(w, \partial W = V) \leqslant \varepsilon$ for all $w \in W$.

Observe that the filling radius of a noncompact manifold V may be infinite. For example, the Euclidean space $\mathbf{R}^n$ has $\operatorname{Fill\,Rad} \mathbf{R}^n = \infty$ for all $n = 1, 2, \cdots$ (see §4.5.D for additional examples).

On the other hand, the argument of Theorem 4.4.A extends our main filling inequality to all complete manifolds V,

$$\operatorname{Fill\,Rad} V \leqslant (n+1) n^n \sqrt{(n+1)!}\, (\operatorname{Vol} V)^{1/n}, \tag{4.3}$$

for all complete manifolds V. In particular, every manifold of finite volume $\operatorname{Vol} V < \infty$ also has Fil Rad $< \infty$. Moreover, there exists a filling W of V (for $\operatorname{Vol} V < \infty$) with the boundary $\partial W = V$, for which

$$\operatorname{dist}(w, \partial W) \to 0, \text{ for } w \to \infty. \tag{4.4}$$

4.5. The filling radius and the injectivity radius. Let X be a complete Riemannian manifold, whose injectivity radius is everywhere greater than

$R_0 > 0$. Take $n + 2$ points $x_0, \cdots, x_{n+1}$ in X such that

$$\operatorname{dist}(x_i, x_j) \leqslant \delta \leqslant R_0/n + 2, \quad \text{for } i, j = 0, \cdots, n+1,$$

and observe that there is a *canonical* geodesic $(n + 1)$-dimensional simplex in X with the vertices $x_0, \cdots, x_{n+2}$. Namely, one first joins x_0 and x_1 by a minimal geodesic segment $\Delta_1(x_0, x_1)$ in X. Then $\operatorname{dist}(x, x_2) \leqslant \frac{3}{2}\delta$ for all $x \in \Delta_1(x_0, x_1)$, and so there is a geodesic cone, say $\Delta(x_0, x_1, x_2)$, from x_2 over $\Delta_1(x_0, x_1)$. As $\operatorname{dist}(x, x_3) \leqslant 2$ for all $x \in \Delta_2(x_0, x_1, x_2)$, one can take for Δ_3 the geodesic cone from x_4 over Δ_2 and so on.

Now let V be a closed n-dimensional submanifold in X, whose filling radius (relative to the induced metric) is at most $\delta/2$. Then we argue as in Lemma 1.2.B, and extend the inclusion map $V \to X$ to a continuous map of a filling chain $c \in L^\infty(V)$ (which spans V without distance $\delta/2$ from $V \subset L^\infty(V)$) to X. Thus we get

4.5.A. Lemma. *If*

$$\text{Fill Rad } V < (\text{Inj Rad } X)/2(n+2),$$

then the submanifold $V \subset X$ is homologous to zero in X.

4.5.B. Corollary. *If*

$$\text{Inj Rad } X \geqslant 2(n+2)(n+1)C_n(\text{Vol } V)^{1/n}, \qquad \textit{for } C_n \leqslant n^n\sqrt{(n+1)!}\,,$$

then the submanifold V bounds in X.

This corollary for $V = X$ reduces to (a non-sharp version of) *Berger's* isembolic inequality

$$\text{Inj Rad } V \geqslant \text{const}_n(\text{Vol } V)^{1/n}.$$

The argument of Lemma 1.2.B, by which we have just proved the above Lemma 4.5.A, easily generalizes to a purely homology-theoretic content and then we obtain an upper bound on Fill Rad V in term of the local homological invariants of V. Namely, for a given number $C \geqslant 1$, we introduce the following "radius" $R_0 = R_0(V, C)$ as the upper bound of the real numbers $R > 0$ with the following property. For every pair of concentric balls in V of radii R and CR, the inclusion homomorphisms on the cohomology groups with $\mathbf{Z}_2$-coefficients,

$$H_i(B(R); \mathbf{Z}_2) \to H_i(B(CR); \mathbf{Z}_2),$$

vanishes for $i = 1, \cdots, n = \dim V$. This "radius" bounds from below the filling radius of V and thus the volume of V as follows.

4.5.C. Theorem. *Let V be a complete Riemannian manifold of finite volume, for which $R_0(V, C) > 0$. Then V is compact and*

$$\operatorname{Vol} V \geqslant (n^2C)^{-(n+1)^2}[R_0(V, C)]^n.$$

4.5.C′. Remark. One obtains, by the argument of §4.4.B, the following more precise result for *open complete* manifolds V, for which $R_0(V, C) > 0$.

The volume of the concentric balls $B(R)$ in V around a fixed point satisfy for $R \to \infty$

$$\liminf R^{-1} \operatorname{Vol} B(R) \geqslant \left(C(n-1)^2\right)^{-n^2}[R_0(V, C)]^{n-1}.$$

4.5.D. Geometrically contractible manifolds. The above considerations take a simpler form with the following definition.

A complete manifold V is said to be *geometrically contractible* if there exists a positive function $R(\rho) = R_V(\rho)$ for $\rho \in [0, \infty)$ such that every ball in V of radius ρ is contractible within the concentric ball of radius $\rho + R(\rho)$.

Examples. (a) The space $\mathbf{R}^n$ is geometrically contractible.

(b) Let V be a complete manifold and let the isometry group $Is = Is(V)$ be *co-compact* on V, that is, the quotient V/Is be a compact space. Then the manifold V is geometrically contractible if and only if it is contractible. The proof is obvious. However, the structure of the relevant "contractibility" function $R_V(\rho)$ may be quite complicated. Indeed, if V is the isometric universal covering of a compact aspherical manifold V_0, then the growth rate of the (minimal) "contractibility" function $R_V(\rho)$, $\rho \to \infty$, reflects the solvability degree of the word problem in the fundamental group $\pi_1(V_0)$. In particular if the word problem in the group $\pi_1(V_0)$ is unsolvable, then the function $R(\rho)$ grows faster, as $\rho \to \infty$, than any recursive function.

(c) Let V be a leaf of a foliation $\mathfrak{F}$ in a compact manifold W. We equip V with the Riemannian metric induced from some Riemannian metric in W. If the manifold V is not *geometrically contractible*, then there obviously exists another leaf V' of $\mathfrak{F}$, which is contained in the closure of V and is *not topologically contractible*. Therefore if *all* leaves of $\mathfrak{F}$ are topologically contractible, then they also are geometrically contractible.

Now the proof of Theorem 4.5.C also yields the following.

4.5.D′. Theorem. *Every complete geometrically contractible manifold V has*

$$\operatorname{Fill} \operatorname{Rad} V = \infty,$$

and therefore

$$\operatorname{Vol} V = \infty.$$

as well.

4.5.E. Geometrically essential manifolds. We say that a complete manifold X is *geometrically aspherical* if the universal covering $\tilde{X}$ of X is geometrically contractible.

Example. Every complete manifold of nonpositive sectional curvature is *geometrically aspherical.* A complete manifold V is called *geometrically essential* if there exists a proper (uniformly!) Lipschitz map of V into some geometrically aspherical manifold X such that the fundamental class of V goes to a nonzero homology class of X with noncompact supports, $f_*[V] \neq 0$. If the class $f_*[V]$ is not homologous to an ordinary cycle in X (i.e., with a compact support), then we say that V is *geometrically essential at infinity.* For example, every complete *noncompact* geometrically aspherical manifold is geometrically essential at infinity.

Now the argument of Theorem 4.5.C and inequalities (4.3) and (4.4) of §4.4.C lead to the following.

Geometrical isosystolic inequality. *Let V be a complete geometrically essential manifold of dimension n. Then*

$$\operatorname{sys}_1 V \leq 6(n+1)n^n\sqrt{n!}\,(\operatorname{Vol} V)^{1/n}.$$

Furthermore, if V is geometrically essential at infinity and $\operatorname{Vol} V < \infty$, *then*

$$\operatorname{sys}_1 V = 0.$$

A counterexample. Let V be the isometric product of the unit circle S^1 by a surface V_0, which is homeomorphic to $\mathbf{R}^2$ and has Area $V_0 < \infty$. Then $\operatorname{sys}_1 V = 2\pi$, no matter how small the volume $\operatorname{Vol} V = 2\pi$ Area V_0 is. This happens because the (aspherical) manifold $V = V_0 \times S^1$ is not geometrically aspherical.

5. Short geodesic in surfaces

Let V be a surface with a complete Finsler metric. We call by *area* the mass* measure on V (see §4.1), and recall the coarea formula

$$\text{(5.1)} \qquad \operatorname{Area} B_v(R) \geq \int_0^R \operatorname{length} \partial B_v(r)\,dr,$$

for balls $B_v(R)$ of radius R. The length of the boundary $\partial B(r)$ is understood as the one-dimensional Hausdorff measure. In fact, one may assume (using an appropriate approximation of the function dist$(v,-)$) the boundaries $\partial B(r)$ to be piecewise smooth for all r, and to be some unions of simple closed curves for almost all $r \in [0,\infty)$. We always make such an assumption whenever we need it.

The principal isosystolic inequalities for surfaces V are proven below on the basis of the formula (5.1) alone. Unfortunately, this direct elementary approach does not generalize to manifolds of dimension ≥ 3.

5.1. The height function and the areas of balls. We denote by $\operatorname{sys}(V, v)$ the length of the shorter noncontractible loop γ in V with the base point v, and by $\operatorname{sys}'(V, v) \geq \operatorname{sys}(V, v)$ the length of the shortest loop which is not homologous to zero mod 2. Observe that the number $\operatorname{sys}(V, v)$ (respectively $\operatorname{sys}'(V, v)$) is the upper bound of those $R > 0$ for which the ball $B_v(R/2)$ is contractible (homologous to zero) in V, where "homologous to zero" means the vanishing of the inclusion homomorphism $H_1(B_v(R/2); \mathbf{Z}_2) \to H_1(V; \mathbf{Z}_2)$.

If a ball B in V has some contractible boundary components, we fill in every such component of ∂B by an open 2-cell in V (which is unique, unless $V \approx S^2$), and denote by B^+ the union of B with these cells.

A loop γ in V at v is said to be *minimal* if it is not homomorphic to a shorter loop at v. We call γ systolic (respectively, homologically systolic) at v if length $\gamma = \operatorname{sys}(V, v)$ (respectively, length $\gamma = \operatorname{sys}'(V, v)$).

A closed curve γ in V is said to be *minimal* (homologically minimal) if it is not freely homotopic (not homologous) to a shorter curve. We say γ is (respectively, *homologically*) *systolic* if

$$\operatorname{length} \gamma = \operatorname{sys} V \overset{\text{def}}{=} \inf_{v \in V} \operatorname{sys}(V, v),$$

(respectively, if length $\gamma = \operatorname{sys}'(V)$).

A complete geodesic γ in V, periodic ($\approx S^1$) or infinite ($\approx \mathbf{R}^1$), is said to be *straight* if the distance between every two points v_1 and v_2 in $\gamma \subset V$ equals the length of the shortest segment of γ between v_1 and v_2.

If V is a compact orientable surface, then the left of every minimal geodesic γ in V to the universal covering $\tilde{V}$ of V is a *straight* geodesic $\tilde{\gamma}$ in $\tilde{V}$ (see [61]). This is also true for *two-sided* minimal geodesics in compact *non-orientable* surfaces. the "two-sided" condition means the vanishing of the first Stiefel-Whitney classes on γ, i.e., $w_1[\gamma] = 0$.

Observe that homologically systolic geodesics in V are straight. Moreover the homology group $H_1(V, \mathbf{Z})$ can be generated by (the classes of) homologically minimal straight geodesics. A particular basis in H_1 can be obtained by induction as follows. We take for γ_i, $i = 1, \cdots, q = b_1(V) = \operatorname{rank} H_1(V)$, the shortest geodesic in V which is not homologous to any integral combination of the geodesics γ_j for $j = 1, \cdots, i - 1$. We call such a system of geodesics *a short basis* in $H_1(V)$.

Now let γ be either a closed curve $\gamma: S^1 \to V$ or an infinite curve $\gamma: \mathbf{R} \to V$. We define *the tension* tens γ as the upper bound of those numbers $\delta > 0$ such that there exists a homotopy γ_t of $\gamma = \gamma_0$, which diminishes the length of γ by δ. This homotopy of an *infinite* curve γ is assumed to be *compact*; it keeps the map $\gamma\colon \mathbf{R} \to V$ fixed outside some compact interval in $\mathbf{R}$. Then the difference $\delta = \text{length } \gamma_0 - \text{length } \gamma_1$ is correctly defined, even if the curve $\gamma = \gamma_0$ has infinite length.

Take a point $v \in V$ and say that a curve γ which passes through v is *admissible* if either γ is a noncontractible closed curve: $\gamma\colon S^1 \to V$, or γ is infinite, i.e., $\gamma\colon \mathbf{R}^1 \to V$, where $\gamma(0) = v$, and both ends of γ, which are the restrictions

$$\gamma \mid \mathbf{R}_+ : \mathbf{R}_+ \to V \text{ and } \gamma \mid \mathbf{R}_- : \mathbf{R}_- \to V,$$

have infinite length. We define the *height* $h(v)$ of v as the lower bound of the tensions of all admissible curves passing through v.

Examples. (a) If V is homeomorphic to S^2, then $h(v) = +\infty$ for all $v \in V$.

(b) Let V be homeomorphic to $\mathbf{R}^2$. Then $h(v) \equiv +\infty$ if and only if there are no straight infinite geodesics in V.

(c) If the fundamental group $\pi_1(V)$ is nontrivial, then the function $h(v)$ is finite at every point v in V. In fact, $h(v) \leqslant \operatorname{sys}(V, v) - \operatorname{sys} V$ for all $V \in V$.

The following three properties of the height function are immediate from the definition.

5.1.A. Proposition. *If V is a complete nonsimply connected surface, then*

(a) *every component of the set $h^{-1}(t, \infty)$ for each $t \in [0, \infty)$ is an open topological 2-cell in V;*

(b) *the subset $h^{-1}(0)$ is the union of minimal closed geodesics in V and of those infinite geodesics in V, whose lifts to the universal coverings $\tilde{V}$ of V are straight;*

(c) *the function $h(v)$ is Lipschitz*, i.e.,

$$h(v_1) - h(v_2) \leqslant 2 \operatorname{dist}(v_1, v_2),$$

for all pairs of points v_1 and v_2 in V.

Now we slightly refine an argument of Berger [13] and Hebda [42] in order to get a lower bound on the areas of balls in V. Notice that Hebda restricts himself at that point (for a reason which is hard to understand) to *orientable* surfaces V.

5.1.B. Proposition. *The area of every ball $B_v(R)$ in a complete surface V satisfies the inequality*

$$\text{Area } B_v(R) \geqslant \frac{1}{2}(2R - h(v))^2,$$

for every R in the interval

$$R \in \left[\tfrac{1}{2}h(v), \tfrac{1}{2}\operatorname{sys}(V, v)\right].$$

In particular, the area of a nonsimply connected surface V satisfies

$$\text{Area } V \geqslant \text{Area } B_v(\tfrac{1}{2}\operatorname{sys}(V,v)) \geqslant \frac{1}{2}\operatorname{sys}(V,v)^2 \geqslant \tfrac{1}{2}(\operatorname{sys} V)^2.$$

Proof. If $R < \frac{1}{2}\operatorname{sys}(V, v)$, then the ball $B_v(R)$ is contractible in V, and so the set $B_v^+(R)$ is a topological disk. Thus every admissible curve γ through v hits the boundary $\partial B_v^+(R)$ at a pair of points, which divide this boundary (which may be assumed a simple closed curve) into two segments of lengths l_1 and $l_2 \geqslant l_1$, such that $l_1 + l_2 = \text{length } \partial B_v^+(R)$. Clearly the tension of γ satisfies

$$\operatorname{tens}\gamma \geqslant 2R - l_1,$$

and so

$$\text{length } \gamma B_v^+(R) \geqslant 4R - 2h(v).$$

Therefore

$$\text{Area } B_v(R) \geqslant \int_0^R \text{length}(B_v r)\, dr \geqslant \int_{h(v)/2}^R \text{length } B_v^+(r)\, dr \geqslant 2(2R - h(v))^2.$$

Let us indicate some related estimates from below for the area of a ball $B_v(R)$ in V.

5.1.B′. *Let the point v lie on the straight (closed or doubly infinite) geodesic γ of (finite or infinite) length l.*

(a) *If γ is infinite or a closed homologically minimal geodesic, then*

$$\text{Area } B_v(R) \geqslant 2R^2, \text{ for } R \leqslant \tfrac{1}{2}\operatorname{sys}'(V,v) \leqslant \tfrac{1}{2}l.$$

(b) *If l is a closed geodesic, which is homologous to zero, then*

$$\text{Area } B_v(R) \geqslant 2R^2, \quad \text{for } 0 \leqslant R \leqslant l/4,$$

$$\text{Area } B_v(R) \geqslant 2R(l-R) - \tfrac{1}{4}l^2, \quad \text{for } l/4 \leqslant R \leqslant l/2.$$

Proof. In eihter case the geodesic γ hits the boundary $\partial B_v(R)$ at two points which necessarily lie in one compact of the boundary $\partial B_v(R)$, and the length of this component is estimated as above.

5.1.B″. Corollary. *Every straight homologically minimal geodesic γ has*

$$\text{length } \gamma \leqslant 2 \text{ Area } V/\operatorname{sys}'(V).$$

Indeed, the area of V can be estimated from below by the total area of mutually disjoint balls of radius $\frac{1}{2}\operatorname{sys}'(V)$ with the centers on γ.

Now let σ be a distance minimizing segment between two points v_1 and v_2 in V, so that length $\sigma = \operatorname{dist}(v_1, v_2)$. Take a point $v \in \sigma$ which divides σ into segments of lengths $l_1 = \operatorname{dist}(v, v_1)$ and $l_2 = \operatorname{dist}(v, v_2)$, and consider a ball

$B_v(R)$ whose radius $R \geqslant 0$ satisfies the following two inequalities:

(1) $R \leqslant \frac{1}{2}\mathrm{sys}(V, v)$,

(2) $R \leqslant \frac{1}{2}l_i - \frac{1}{4}h(v_i)$, for $i = 1, 2$.

5.1.C. Proposition. *The above ball $B_v(R)$ has*

$$\text{Area } B_v(R) \geqslant 2R^2.$$

Proof. Take the disk B_v^+ and consider the following two possibilities:

(a) None of the end points v_1 and v_2 is contained in $B_v^+(R)$. Then length $\partial B_v^+(R) \geqslant 4R$.

(b) One of the end points, say v_1, is in $B_v^+(R)$. Then $d = \mathrm{dist}(v_1, \partial B_v^+(R)) \geqslant l_1 - R$ and

$$h_1(v) \geqslant 2d - \tfrac{1}{2} \text{ length } \partial B_v^+(R).$$

Now using (2) we have again length $\partial B_v^+(R) \geqslant 4R$.

5.1.C′. Corollary (*Compare Corollary* 5.1.B″). *The diameter of the subset $h^{-1}[0, t] \subset V$ satisfies for every $t \geqslant 0$,*

$$\text{Diam } h^{-1}[0, t] \leqslant t + \mathrm{sys}\, V + \text{Area } V/\mathrm{sys}\, V.$$

5.1.D. Minimal graphs in V. We define the *tension* of a finite graph (i.e., a one-dimensional subcomplex) Γ in V as the upper bound of those numbers $\delta \geqslant 0$, for which there exists a continuous map $f\colon \Gamma \to V$, homotopic to the inclusion map $i\colon \Gamma \to V$, whose image has

$$\text{length } f(\Gamma) \leqslant \text{length } \Gamma - \delta.$$

If V is compact, then there exists a map f homotopic to $i\colon \Gamma \hookrightarrow V$, whose image $\Gamma' = f(\Gamma) \subset V$ is a *minimal* graph, i.e., tens $\Gamma' = 0$. Every component of Γ' is either a closed geodesic in V or a finite union of vertices joined by simple geodesic segments and loops, with exactly three geodesics approaching every nonisolated vertex. Furthermore, if the original graph Γ contains the one-skeleton of some cell decomposition of V, then the complement $V \setminus \Gamma'$ is a topological 2-cell, whose lifts to the universal covering $\tilde{V}$ of V are fundamental polygons for the deck transformation group on $\tilde{V}$.

The above arguments imply that a ball $B_v(R)$, whose center v is contained in a connected noncontractible graph $\Gamma \subset V$, satisfies

$$\text{Area } B_v(R) \geqslant (2R - h)^2,$$

for $\frac{1}{2}h \leqslant R \leqslant \frac{1}{2}\mathrm{sys}(V, v)$ and $h =$ tension(Γ).

5.2. A sharp isosystolic inequality for surfaces. Let V be a closed surface with an infinite fundamental group (i.e., V is not homeomorphic to S^2 or to $\mathbf{R}P^2$). Let γ be a closed minimal two-sided geodesic in V.

5.2.A. Theorem. *For every $R \in [0, \frac{1}{2}\operatorname{sys} V]$ there exists a ball $B_v(R)$ in V with the center v on $\gamma \subset V$ such that*

$$\operatorname{Area} B_v(R) \geqslant 3R^2.$$

Proof. It suffices to show that the average length of the boundary of the disk $B_v^+(R)$ over $v \in \gamma$ is at least $6R$:

$$\int_\gamma \operatorname{length} B_v^+(R)\, d\gamma \geqslant 6R.$$

We show this by applying the following lemma to a lift of γ to a straight (!) infinite geodesic $\tilde{\gamma}$ in the universal covering of V.

5.2.A′. Lemma. *Let W be a complete simply connected surface, and let w_i, $i = \cdots -1, 0, 1, \cdots$, be a double infinite sequence of points in W such that*

$$\operatorname{dist}(w_i, w_j) = R|i - j|,$$

for all $-\infty < i, j < \infty$. Then the average length of the boundary $B_{w_i}^+(R)$ is at least $6R$:

$$\liminf_{N \to \infty} N^{-1} \sum_{i=1}^{N} \operatorname{length} \partial B_{w_i}^+(R) \geqslant 6R.$$

Proof. The points w_i lie on an infinite straight geodesic $\tilde{\gamma}$ in W. Each boundary $\partial B_{w_i}^+(R)$ has two segments of lengths $\geqslant 2R$ in the two neighboring disks $\partial B_{w_{i-1}}^+(R)$ and $\partial B_{w_{i+1}}^+(R)$. Furthermore there are two segments of $\partial B_{w_i}^+(R)$, say σ_i and σ_i', which lie on the boundary of the union $\bigcup_{-\infty}^{\infty} B_{w_i}^+(R)$, on both sides of $\tilde{\gamma}$. As the geodesic $\tilde{\gamma}$ is straight, the average length of these segments on either side of $\tilde{\gamma}$ is at least R, and the lemma follows.

5.2.B. Corollary. *Every compact surface with an infinite fundamental group has*

$$\operatorname{Area} \geqslant \tfrac{3}{4}(\operatorname{sys})^2. \tag{5.2}$$

5.2.B′. Remarks. This result for *Finsler tori* is due to Zaustinsky [78]. In fact, his inequality for the hyper-Euclidean volume (area) $\operatorname{Vol} \leqslant (2/\sqrt{3})\operatorname{mass}^*$ (see §4.1) reduces (5.2) to Loewner's inequality $\operatorname{Vol} \geqslant (\sqrt{3}/2)(\operatorname{sys})^2$ for the volume (area) of *Riemannian* metrics on tori (see §§0.3, 5.5). The extremal tori for which (5.2) becomes an equality are Finsler flat with the regular hexagon for the unit ball in the universal covering (see [78]).

The proof of Theorem 5.2.A shows that surfaces of genus > 1 have Area $\geqslant (\frac{3}{4} + \varepsilon)(\operatorname{sys})^2$, for $\varepsilon > 0.01$, while every Finsler metric on $\mathbf{R}P^2$ satisfies Area $\geqslant (\frac{1}{2} + \varepsilon')(\operatorname{sys})^2$. In fact, Zaustinsky's inequality and Pu's theorem (see §§0.3, 5.5) show $\varepsilon' > (\sqrt{3}/\pi) - \frac{1}{2}$.

5.3. Isosystolic inequalities for surfaces of large genus $\to \infty$. Let $\gamma_1, \cdots, \gamma_q$, for $q = b_1(V) = 2 - \chi(V)$, be a short basis (see §5.1) in a closed surface V. Let l_i, $i = 1, \cdots, q$, denote the lengths of the geodesics γ_i. Observe that l_1 equals the homological systole sys′(V). Let the union $\Gamma = \bigcup_i \gamma_i$ be covered by N balls B_j, $j = 1, \cdots, N$, in V of radius $R < \frac{1}{4}\mathrm{sys}' V$. Then every curve γ_i is homologous to a path of geodesic segments between the centers of these balls B_j. Therefore $b_1(V) \leqslant M - N + b_0'$, where M denotes the number of intersections between the pairs of the balls B_j, and b_0' the number of connected components of the union $\bigcup_j B_j$.

5.3.A. Proposition. *The first Betti number* $b_1(V)$ *satisfies*

$$b_1(V) \leqslant \tfrac{1}{2}(N-1)(N-2), \tag{5.3}$$

for $N \leqslant 32$ *Area* $V/(\mathit{sys}' V)^2$, *so that*

$$\text{Area } V/(\mathrm{sys}' V)^2 > \sqrt{b_1(V)}\,/16\sqrt{2}\,.$$

Proof. Take a maximal system of disjoint balls of a fixed radius $\frac{R}{2} < \frac{1}{8}\mathrm{sys}' V$ with the centers in Γ. The area of each ball is at least $\frac{1}{2}R^2$ (see §5.1.B′) and so the total number N of them is at most 2 Area V/R^2. The concentric balls of radius R cover the set Γ so that

$$b_1(V) \leqslant \tfrac{1}{2}N(N-1) - N + 1 = \tfrac{1}{2}(N-1)(N-2)$$

Remark. One may take only those balls which lie in the complement of some (contractible!) ball B_0 of radius $\frac{3}{8}\mathrm{sys}\, V$. There is such a ball B_0 of area $= A_0 \geqslant \frac{27}{64}(\mathrm{sys}\, V)^2$, (see Theorem 5.2.A) and so one sharpens the inequality (5.3) by substituting

$$N' = 32(\text{Area } V - A_0)/\,(\mathrm{sys}' V)^2 \text{ for } N.$$

The following theorem sharpens the inequality (5.3) for $b_1(V) \to \infty$.

5.3.B. Theorem. *The first Betti number satisfies*

$$b_1(V) \leqslant 5^{3\sqrt{\log A'}} A', \tag{5.4}$$

where $\log = \log_5$ *and*

$$A' = 40 \text{ Area } V/\,(\mathrm{sys}' V)^2.$$

Therefore, for any given $\theta > 0$,

$$\text{Area } V/\,(\mathrm{sys}' V)^2 \geqslant \mathrm{const}_\theta[b_1(V)]^{1-\theta},$$

for some positive constant const_θ *which depends only on* θ.

Proof. A ball $B = B_v(R)$, for $v \in \Gamma$ and $R < R_0 = \frac{1}{8}\mathrm{sys}' V$, is said to be *α-admissible* for some given $\alpha > 1$ (which will be specified later on), if

(1) Area $5B \leqslant \alpha$ Area B, where $5B$ denotes the concentric ball $B_v(5R)$,

and

(2) every larger concentric ball $B' = B_v(R')$, for $R < R' \leqslant R_0$, has α Area $B' \leqslant$ Area $5B'$.

Let us estimate from below the area of an admissible ball $B = B_v(R)$. Take the integer $r \geqslant 0$, for which

$$5^{-r}R_0 \leqslant R \leqslant 5^{-r+1}R_0,$$

and observe that

$$A = \text{Area } V \geqslant \alpha^r \text{ Area } B \geqslant 2\alpha^r R^2.$$

Hence

$$A \geqslant 2\alpha^r 5^{-2r} R_0^2,$$
$$r < r_0(\alpha) = (\log A - 2\log R_0)/(\log\alpha - 2),$$
$$\text{Area } B > A(\alpha) = 2\big(5^{-2r_0(\alpha)}R_0^2\big).$$

Now we construct a maximal system of disjoint admissible balls $B_1, \cdots, B_N$ by taking for each B_j, $j = 1, \cdots, N$, an admissible ball of the greatest radius R_j, which does not intersect the balls $B_{j'}$ for $j' < j$.

Observe that the concentric balls $2B_j$ cover the subset $\Gamma \subset V$, and let us estimate the number M of double intersection between the balls $2B_j$. If a ball $2B_j$ meets some ball $2B_{j_k}$, for $j_k > j$ and $k = 1, \cdots, m_j$, then the concentric ball $5B_j$ contains the balls B_{j_k} for $k = 1, \cdots, m_j$. As the ball B_j is admissible,

$$\text{Area } B_j \geqslant \alpha^{-1} \sum_{k=1}^{m_j} \text{Area } B_{j_k},$$

we have

$$A = \text{Area } V \geqslant \sum_j \text{Area } B_j \geqslant \alpha^{-1} \sum_{j=1}^{N} M_j A(\alpha) \geqslant \alpha^{-1} M A(\alpha),$$
$$M \leqslant \alpha A / A(\alpha).$$

Let us take α such that

$$\log\alpha = 2 + \sqrt{\log A - 2\log R_0}, \quad \text{for } R_0 = \tfrac{1}{8}\text{sys}' V.$$

Then a straightforward calculation shows that

$$M \leqslant 5^{3\sqrt{\log A'}} A',$$

which implies the inequality (5.4).

5.3.B′. Remark. The above covering argument also yields the following combinatorial isosystolic inequality for finite graphs.

Let a connected graph Γ have p vertices and $q \geqslant p$ edges. Let $s = \operatorname{sys}\Gamma$ be the length of the shortest (nontrivial) cycle in Γ. Then

$$b_1(\Gamma) = q - p + 1 \leqslant 5^{3\sqrt{\log A'}} A',$$

for $A' = 40q/s$.

5.3.C. Upper bounds for the lengths $l_i = $ length γ_i, for $i > 1$. Proposition 5.1.B″ shows that

$$l_i \leqslant 2 \text{ Area } V/l_1 = 2 \text{ Area } V/\operatorname{sys}' V,$$

for $i = 1, \cdots, q = b_1(V)$. In fact, most geodesics γ_i are much shorter than that.

5.3.C′. Proposition. *The number q_l of the geodesic γ_i, which have*

$$l_i = \operatorname{length} \gamma_i \geqslant l,$$

satisfies for every $l \geqslant 4l_1$

$$q_l \leqslant \tfrac{1}{2} N_l (N_l - 1),$$

for some number $N_l \leqslant 64$ Area $V/(l_1 l)$.

Proof. Let Γ_l be the union of the geodesics γ_i for which $l_i > l$. Each ball $B_v(R)$, for $v \in \Gamma_l$ and $l_1/2 \leqslant R \leqslant l/2$, has

$$\text{Area } B_v(R) \geqslant \frac{l_1^2}{2} + 2l_1\left(R - \frac{l_1}{2}\right) \geqslant l_1 R.$$

Therefore (compare Proposition 5.3.A) the subset $\Gamma_l \subset V$ can be covered by $N_l \leqslant 64$ Area $V/(l_1 l)$ balls B_j of radius $l/4$. Then every geodesic $\gamma_i \subset \Gamma_l$ is homologous to some combination of geodesics of length $< l$ and a closed path of geodesic segments between the centers of the balls B_j.

5.3.C″. Remark. One may sharpen Proposition 5.3.C′ by using *admissible* balls B_j (compare Theorem 5.3.B).

Now let us estimate the total length of the geodesics γ_i.

5.3.D. Proposition. *The sum of the lengths of γ_i satisfies*

$$L = \sum_{i=1}^{q} l_i \leqslant 2^7 3^5 (\text{Area } V)^3 / l_1^5.$$

Proof. Let $W = V \times V \times V$ and let $\operatorname{dist}(w, w') = \max_{\nu=1,2,3} \operatorname{dist}(v_\nu, v'_\nu)$ for $w = (v_1, v_2, v_3)$ and $w' = (v'_1, v'_2, v'_3)$ in W.

Then we consider the triples $w = (v_1, v_2, v_3) \in W = V \times V \times V$, for which the points v_ν, $\nu = 1, 2, 3$, lie on some geodesic γ_i, $i = 1, \cdots, q$, such that each γ_i is divided by these points into three segments of lengths l_{i_ν} which satisfy for $\nu = 1, 2, 3$

$$2l_{i\nu} + \tfrac{1}{3} l_1 \leqslant l_i.$$

If two such triples w and w' lie on different geodesics of our short basics, then $\operatorname{dist}(w, w') \geqslant \frac{1}{6} l_1$, since none geodesic l_i is decomposable (in $H_1(V)$) into shorter curves. On the other hand, each geodesic γ_i supports some triples w^μ, $\mu = 1, \cdots, m_i$, such that $\operatorname{dist}(w^\mu, w^{\mu'}) \geqslant \frac{1}{6} l_1$, for $\mu \neq \mu'$ and $m_i \geqslant 12 l_i / l_1$. Thus we get at least $M = 12L/l_1$ triple w^j in W, for which $\operatorname{dist}(w^j, w^{j'}) \geqslant \frac{1}{6} l_1$ for $1 \leqslant j \neq j' \leqslant M$. The products of balls of radius $R = \frac{1}{12} l_1$ around the points v_ν^j, $\nu = 1, 2, 3$, are disjoint in W for $j \neq j'$, and so

$$\operatorname{Vol} W = (\operatorname{Area} V)^3 \geqslant M(2R^2)^3 \geqslant L l_1^5 / (2^7 3^5).$$

Finally, we show that many geodesics γ_i are almost as short as γ_1.

5.3.E. Theorem. *For every $\theta > 0$ there exists a constant* $\operatorname{const}'_\theta$ *such that the first $m - 1$ geodesics γ_i, $i = 1, \cdots, m - 1$, for some $m \geqslant$* $\operatorname{const}'_\theta [b_1(V)]^{1-\theta}$, *have*

$$(\text{length}\, \gamma_i)^2 \leqslant \operatorname{Area} V / [b_1(V)]^{1-\theta}. \tag{5.5}$$

Proof. Let us by induction construct a sequence of auxiliary surfaces V_i, $i = 1, \cdots, m - 1$, and a sequence of geodesics $\gamma'_i \subset V_i$. Take $V_1 = V$ and $\gamma'_1 = \gamma_1$. We obtain V_{i+1} by first cutting V_i along the geodesic γ'_i, and then attaching two round hemispheres $S^2_+(l'_i)$, with the equators of length $l'_i =$ length γ'_i, to the manifold $V_i \backslash \gamma'_i$. If the geodesics γ'_i is one-sided, then we attach one hemisphere $S^2_+(2l'_i)$. We take a homologically systolic geodesic in V_{i+1} for γ'_{i+1}.

We apply Theorem 5.3.B to the surfaces V_i and estimate by induction the length of γ'_i and thus the areas of V_{i+1}. A straightforward calculation shows that the lengths of the first $m - 1$ geodesics γ'_i do satisfy (5.5), and as $l_i \leqslant l'_i$, we obtain the inequality (5.5) for $l_i =$ length γ_i.

5.4. Pairs of short loops in surfaces V of negative Euler characteristic. Take a systolic loop γ in V at a point $v_0 \in V$ and let γ' be another loop at v_0, which does not normalize the cyclic subgroup $Z(\gamma)$ generated by the (homotopy class of) loop γ in the group $\pi_1(V, v_0)$, that is, $[\gamma'][\gamma][\gamma']^{-} \neq Z(\gamma)$. Then obviously there exist (compare [37, p. 76]) two independent systolic loops γ_1 and γ_2 at some point $v \in \gamma' \in V$ such that γ_1 is freely homotopic to γ. Here "independent" means that the subgroup generated by γ_1 and γ_2 in $\pi_1(V, v)$ is a free group of rank 2.

5.4.A. Theorem. *If $\chi(V) < 0$, then there exists a pair of independent systolic loops γ_1 and γ_2 at some point $v \in V$ such that*

$$\text{length}\, \gamma_1 = \text{length}\, \gamma_2 = \operatorname{sys}(V, v) \leqslant \sqrt{2 \operatorname{Area} V}.$$

Proof. Start with a systolic geodesic γ and take for γ' any minimal geodesic which intersects γ and does not normalize $Z(\gamma)$.

5.4.A′. Corollary. *The balls $B_{\tilde V}(R)$ in the universal covering $\tilde V$ of V with the center $\tilde v$ over v satisfy for $R \geqslant s = \operatorname{sys}(V, v)$*

$$\text{Area } B_{\tilde v}(R) \geqslant \tfrac{1}{2}s^2 3^{(R-s)/2}.$$

Indeed, the translates of the ball $B_{\tilde v}(s/2)$ (which has area $\geqslant \frac{1}{2}s^2$; see §5.1.B′) under the free (!) subgroup of deck transformations, generated by γ_1 and γ_2, provide so much area.

Let us sharpen Theorem 5.4.A for surfaces of large genus $\to \infty$.

5.4.B. Theorem. *There exist two independent loops of length $< \varepsilon$ at some point $v \in V$, for $\varepsilon \leqslant 6l_0$ and*

$$l_0 = C_\theta \sqrt{\text{Area } V/[b_1(V)]^{1-\theta}},$$

where θ is an arbitrary positive number, and C_θ is a positive constant which depends only on θ.

Proof. Let $\chi(V) < 0$ and let γ be an arbitrary noncontractible curve in V. Then there exists a minimal loop $\bar\gamma$ at some point v in the set $h^{-1}(0) \subset V$ (see §5.1), which is freely homotopic to γ and has a prescribed length $l \geqslant$ length γ. In fact, one can find such a loop $\bar\gamma$ at some point v on any minimal geodesic γ', which intersects γ and is independent of γ (in $\pi_1(V, v_0)$, $v_0 \in \gamma \cap \gamma'$).

Let γ_i, $i = 1, \cdots, m-1$, be the short loops provided by Theorem 5.3.E, and let $\bar\gamma_i$ be the corresponding loops of length l_0 at some points $v_i \in h^{-1}(0) \subset V$. Let V be orientable. If there is no loop $\tilde\gamma_i$ at any point v_i, which has length $\leqslant 3l_0$ and is independent of $\bar\gamma_i$, then we conclude that

(a) every loop $\bar\gamma_i$, $i = 1, \cdots, m-1$, is systolic, that is, length $\bar\gamma_i = \operatorname{sys}(V, v_i)$;

(b) the balls $B_{v_i}(l_0/2)$ do not intersect for $i = 1, \cdots, m-1$.

Therefore

$$\text{Area } V \geqslant (m-1)l_0^2/2 \geqslant \tfrac{1}{2}\left(\text{const}'_\theta[b_1(V)]^{1-\theta} - 1\right)C_\theta^2\left(\text{Area } V/[b_1(V)]^{1-\theta}\right),$$

and so

$$C_\theta \leqslant \sqrt{2}\left(\text{const}'_\theta - [b_1(V)]^{\theta-1}\right)^{-1/2}.$$

As the right-hand side of the last inequality is positive for $b_1(V) \to \infty$, we arrive at a contradiction, and thus obtain the theorem for orientable surfaces V. The nonorientable case now follows by passing to the oriented double covering.

5.5. Conformal isosystolic inequalities. Let Γ be some class (i.e., a family) of curves γ in a surface V with a Riemannian metric g. One defines

$$\text{length}_g \Gamma = \inf_{\gamma \in \Gamma} \text{length}_g \gamma.$$

Then one considers all those conformal metrics $g' = \varphi^2 g$ on V, for which

$$\mathrm{Area}(V, g') = \int_V \varphi^2\, dv \leqslant 1,$$

for $dv = d(\mathrm{Area}_g)(v)$, and one defines the *conformal* (or extremal, see [48]) *length* of Γ as

$$\sup_{g'} \mathrm{length}_{g'} \Gamma = \sup_{\varphi} \left(\inf_{\gamma \in \Gamma} \int_\gamma \varphi(v)\, d\gamma \right),$$

over the above conformal metrics $g' = \varphi^2 g$, where $d\gamma = d(\mathrm{length}_g)(v)$ on the curves γ in V.

The conformal length clearly is a conformal invariant of (V, g) and satisfies the inequality

$$\text{conf length} \geqslant \text{length}/\sqrt{\mathrm{Area}\, V}\,.$$

If one enlarges the class Γ, then the conformal length (as well as the ordinary length) only may diminish. For example, the conformal length of the *homology* class of a closed curve $\gamma \subset V$ even may be strictly less than the conformal length of the *homotopy* class of γ.

The following method is commonly used to obtain an upper bound for the conformal length of Γ (see [47]).

Let $d\mu$ be some measure on Γ of the total mass $M < \infty$. Consider the "product measure" $d\mu d\gamma$ on the union $\hat{\Gamma} = \bigcup_{\gamma \in \Gamma} \gamma$, and suppose that the push forward of this measure to V under the map $I\colon \hat{\Gamma} \to V$, which is "the union" of the inclusions $\gamma \in V$, $\gamma \in \Gamma$, has a density function of class L^2, called $\mu_* = \mu_*(v)$, relative to the Riemannian measure (area) dv in $V = (V, g)$, that is, $I_*(d\mu\, d\gamma) = \mu_*\, dv$. Now for any conformal metric $g' = \varphi^2 g$ we have

$$\begin{aligned}\mathrm{length}_{g'} \Gamma &\leqslant M^{-1} \int_\Gamma d\mu \int_\gamma \varphi(v)\, d\gamma = M^{-1} \int_V \mu_*(v) \varphi(v)\, dv \\ &\leqslant M^{-1} \|\mu\|_{L_2} \|\varphi\|_{L_2} \leqslant M^{-1} \|\mu_*\|_{L_2}.\end{aligned}$$

Hence the conformal length satisfies

$$\text{Conf length}\, \Gamma \leqslant M^{-1} \|\mu_*\|_{L_2}. \tag{5.6}$$

5.5.A. Examples. (a) Let V be the Cartesian product of the circle $S^1 = S^1(l)$ of length l by the interval $[0, t_0]$. Let Γ be the class of closed curves which are homotopic to the kth multiple kS^1 of $S^1 = S^1 \times 0 \subset S^1 \times [0, t_0]$. There is a natural measure $d\mu$ on Γ which is supported on the circles $kS^1 \times t$, $t \in [0, t_0]$, that is, $d\mu = dt$. The function μ_* for this measure is constant. Namely

$$\mu_* = Mkl_0/\operatorname{Vol} V = t_0 kl/lt_0 = k,$$

and so

$$\operatorname{conf\,length} \Gamma \leqslant t_0^{-1}\sqrt{k^2 \operatorname{Area} V} = k\sqrt{l/t_0}\,.$$

As

$$\operatorname{conf\,length} \Gamma \geqslant \operatorname{length} \Gamma/\sqrt{\operatorname{Area} V} = k\sqrt{l/t_0}\,,$$

we get

$$\operatorname{conf\,length} \Gamma = k\sqrt{l/t_0}\,.$$

(b) *Let the surface V be compact and let G be a compact transitive group of isometries on V. If the class Γ is G-invariant, then*

$$\operatorname{conf\,length} \Gamma = \operatorname{length} \Gamma/\sqrt{\operatorname{Area} V}\,.$$

Proof. Take the normalized Haar measure $d\mu$ on the orbit $G\gamma \subset \Gamma$ of some curve $\gamma \in \gamma$. Then

$$\mu_* \equiv \text{const} = \operatorname{length} \gamma/\operatorname{Area} V,$$

and so

$$\operatorname{conf\,length} \Gamma \leqslant \operatorname{length} \Gamma \leqslant \operatorname{length} \gamma/\sqrt{\operatorname{Area} V}\,,$$

for all $\gamma \in \Gamma$. Hence

$$\operatorname{conf\,length} \Gamma = \operatorname{length} \Gamma/\sqrt{\operatorname{Area} V}\,.$$

(c) *Theorems of Loewner and Pu.* Let g be a Riemannian metric on the torus T. Then g is conformal to a flat metric g_0 on T, which admits a connected transitive group of isometries. Therefore the homotopy class Γ of any closed curve γ in T satisfies

$$\begin{aligned}\operatorname{length}_g \Gamma/\sqrt{\operatorname{Area}(T, g)} \leqslant \operatorname{conf\,length}_g \Gamma &= \operatorname{conf\,length}_{g_0} \Gamma \\ &= \operatorname{length}_{g_0} \Gamma/\sqrt{\operatorname{Area}(T, g_0)}\,.\end{aligned}$$

Hence the inequality $(\text{sys})^2/\text{Area} \leqslant 2/\sqrt{3}$ for g follows from this inequality for g_0 (see §0.3).

In the same way one derives Pu's inequality

$$\text{sys}^2/\text{Area} \leqslant \pi/2,$$

for metrics on $\mathbf{R}P^2$, as they are conformal to an invariant metric of constant curvature.

(d) Let V be a complete locally homogeneous surface of finite area. The principal example is a surface of constant negative curvature.

Let the curves $\gamma \subset V$ of a class Γ lift to the universal covering $\tilde{V}$ of V, such that the lifted class $\tilde{\Gamma}$ of the curves $\tilde{\gamma}$ in $\tilde{V}$ is invariant under the isometries of $\tilde{V}$. Then we conclude as before that

$$\operatorname{conf\,length} \Gamma = \operatorname{length} \Gamma / \sqrt{\operatorname{Area} V}\,.$$

For example, the class $\Gamma = \Gamma(l)$ of the geodesic segments in V of length l has

$$\operatorname{conf\,length} \Gamma = l / \sqrt{\operatorname{Area} V}\,.$$

Corollary. *Let V be a complete surface of constant curvature and finite area. Let f be a conformal homeomorphism of V onto an arbitrary surface V', and let $\tilde{f}$: $\tilde{V} \to \tilde{V}'$ be the induced map between the universal covreings. Then for every $l > 0$ there exists a pair of points v_1 and v_2 in $\tilde{V}$ such that* $\operatorname{dist}(v_1, v_2) = l$ *and*

$$\operatorname{dist}\big(\tilde{f}(v_1), \tilde{f}(v_2)\big) \leqslant l\sqrt{\operatorname{Area} V'} / \sqrt{\operatorname{Area} V}\,.$$

This simple fact was put by Katok [49] into the framework of the ergodic theory. Namely, let γ be an infinite geodesic in $\tilde{V}$, whose points $v \in \gamma$ are parametrized by the length parameter t.

If the map f is uniformly continuous (for example, if V is compact), then for almost all geodesics γ,

$$\limsup_{t \to \infty} t^{-1} \operatorname{dist}\big[\tilde{f}(v(0)), \tilde{f}(v(t))\big] \leqslant \sqrt{(\operatorname{Area} V')/\operatorname{Area} V}\,.$$

Proof. Let $\tilde{d}(v) = \operatorname{dist}(\tilde{v}, \tilde{\gamma})$ and let $\tilde{\gamma}(\alpha)$ be the levels $\tilde{d}^{-1}(\alpha)$, $\alpha \geqslant 0$, which lie on one side of $\tilde{\gamma}$. Take the segments of these curves between two normals to $\tilde{\gamma}$ at the points $v(0)$ and $v(t)$, and let $\tilde{\Gamma}(t, \varepsilon)$ denote the family of those segments for which $0 \leqslant \alpha \leqslant \varepsilon$. This class $\tilde{\Gamma} = \tilde{\Gamma}(t, \varepsilon)$ carries a natural measure $d\mu = d\alpha$, which projects to a measure on the class Γ in V under $\tilde{\Gamma}$. The erogidic properties of the geodesic flow on V imply that almost every geodesic γ is *equidistributed* in V, and hence the function $\mu_*/t\varepsilon$ on V converges, as $t \to \infty$ and $\varepsilon \to 0$, to a constant $\equiv (\operatorname{Area} V)^{-1}$. Therefore

$$\limsup_{\substack{t \to \infty \\ \varepsilon \to 0}} t^{-1} \operatorname{conf\,length} \Gamma(t, \varepsilon) \leqslant (\operatorname{Area}\ V)^{-1/2},$$

and Katok's theorem follows.

5.5.B. Rigidity theorems of Katok and Michel. Let V be a closed Riemannian manifold. Let N_l denote the number of the free homotopy classes of curves in V of length $\leqslant l$ and let

$$\operatorname{hom\,ent} V = \liminf_{l \to \infty} l^{-1} \log N_l.$$

Furthermore let $\tilde{B}_{v_0}(R)$ be the concentric balls in the universal covering $\tilde{V}$ around a fixed point $v_0 \in \tilde{V}$. Put

$$\text{vol ent } V = \lim_{R\to\infty} \left[R^{-1} \log \text{Vol}_{v_0} B(R)\right].$$

Recall (see [24], [56]) that both entropies are equal, hom ent = vol ent, if the manifold V has *negative* sectional curvature. Furthermore, if V has *constant* sectional curvature $-\kappa^2$, then obviously

$$\text{vol ent} = (n-1)\kappa, \quad \text{for } n = \dim V.$$

Main conjecture. Let V_0 be an n-dimensional manifold of constant negative curvature, and let $f\colon V \to V_0$ be a continuous map of degree $d > 0$. Then

$$\frac{\text{hom ent } V}{\text{hom ent } V_0} \geqslant \left(d \frac{\text{Vol } V_0}{\text{Vol } V}\right)^{1/n}, \tag{5.7}$$

as well as

$$\frac{\text{vol ent } V}{\text{vol ent } V_0} \geqslant \left(d \frac{\text{Vol } V_0}{\text{Vol } V}\right)^{1/n}. \tag{5.8}$$

Furthermore, the equality in either of the cases (5.7) and (5.8) would imply that the manifold V also has constant negative curvature and that the map f is homotopic to a d-sheeted covering.

Remark. It is more natural to ask for a stronger version of the inequality (5.8). Namely, the volume entropy of V may be defined *relative to the map f*, that is, with the balls $\tilde{B}_v(R)$ *in the covering which is induced by f from the universal covering* $\tilde{V}_0$ *of* V, rather than with our balls in the (larger) universal covering of V. The corresponding improvement of the inequality (5.7) is also conjectured to be true.

5.5.B′. Theorem (*see* [49]). *The main conjecture is true for* $\dim V = 2$.

Remarks. (a) Katok states and proves his theorem (see [49]) for the map f which is a homeomorphism. Nevertheless, as $n = 2$, this implies the conjecture in the form stated above. However, Katok's argument (which is indicated in Examples 5.5.A) does not seem to imply the stronger version of the conjecture, which is described in the previous remark.

(b) A nonsharp version of (5.8),

$$\frac{\text{vol ent } V}{\text{vol ent } V_0} \geqslant \text{const}_n \left(d \frac{\text{Vol } V_0}{\text{Vol } V}\right)^{1/n},$$

for some universal constant,

$$0 < \text{const}_n < 1,$$

is proven in [32].

(c) If V is a manifold of negative curvature, then the homotopy entropy hom ent is expressible in terms of the numbers N_l' of closed geodesic in V of length $< l$ as $l \to \infty$. Therefore (see [21]), this entropy is a *spectral invariant* of V; it is determined by the eigenvalues of the Laplace operator on V. As the volume of V is also a spectral invariant, we conclude the following corollary of the *rigidity part* of Katok's theorem, which characterizes metric of constant curvature as the extremal metrics for the functional (hom ent) $\sqrt{\text{Area}}$.

Let (V, g_0) *be a closed surface with a metric* g_0 *of constant negative curvature. If a metric* g *of negative curvature on* V *is isospectral to* g_0, *then* g *also has constant curvature*. (Compare [41]).

(d) Katok's theorem also implies the following *minimality property* of the hyperbolic plane $H^2 \subset L^\infty(H^2)$

Let $W \subset H^2$ be a compact connected domain with smooth boundary $V = \partial W$. We denote by dist $| V$ the restriction of the hyperbolic distance to W, and then take another surface with a Riemannian metric (W', g'), which spans $V = \partial W'$ such that $\text{dist}' | V \geqslant \text{dist} | V$ (compare §2.2), where $\text{dist}' = \text{dist}_{g'}$ is defined (like dist in W) as the length of the shortest curve in W' between two points.

Proposition. *If* W' *is homeomorphic to* W, *then* Area $W' \geqslant$ Area W, *and the equality* Area $W' =$ Area W *implies that* W' *is isometric to to* W.

Proof. Following Michel (see [59]) we take a compact surface X of constant curvature such that the universal covering map $p\colon H^2 = \tilde{X} \to X$ is injective on $W \subset H^2$. Then we cut from X the image $p(W) \subset X$, and glue in the surface W'. The resulting closed surface X' is homeomorphic to X. As $\text{dist}' | V \geqslant \text{dist} | V$ for $V = \partial W = \partial W' = \partial(p(W))$, we have

$$\text{vol ent } X' \leqslant \text{vol ent } X,$$

and Katok's theorem applies.

(d′) It seems unlikely that the homeomorphism between W' and W is an essential condition.

(e) The above argument, together with Pu's theorem in place of Katok's theorem, yields the following information on the filling volume (area) of the circle S^1 of length 2.

Proposition. *Let* V *be a disk with a Riemannian metric whose boundary* (*with the induced distance function*) *is isometric to* S^1. *Then*

$$\text{Area } V \geqslant 2\pi,$$

where the equality implies that V *is isometric to the round hemisphere*.

Proof. Identify the opposite points on the boundary $\partial V = S^1$, and apply Pu's theorem to the resulting projective plane.

(e′) If we could prove the above proposition for surfaces V of genus > 0, we would get the equality

$$\operatorname{Fill Vol} S^1 = 2\pi.$$

Now we turn to the following *boundary rigidity problem* which is studied by Michel in [59].

Up to what extent a Riemannian metric on a manifold W with boundary is determined by the restriction of the distance function (on W) to the boundary $V = \partial W$?

Michel points out that for many manifolds W, the total volume $\operatorname{Vol} W$ is determined by the restriction $\operatorname{dist} | \partial W$. Namely, suppose that for every point $v \in V = \partial W$ and every tangent vector $\tau \in T_v(V)$ such that $\|\tau\| < 1$, there exists a unique point $v' = v'(\tau)$ in V with the following two properties:

(1) $\operatorname{grad}_v \operatorname{dist}(v, v') = \tau$ for $\operatorname{dist} = \operatorname{dist} | V$,

(2) there is no point v'' in V different from v and v' such that

$$\operatorname{dist}(v, v'') + \operatorname{dist}(v', v'') = \operatorname{dist}(v, v').$$

In terms of the ambient manifold $W \supset V = \partial W$ these conditions can be loosely expressed by saying that every two points v and v' in V can be joined by at most one geodesic segment inside W.

Now Santalo's formula says (see [68])

$$\operatorname{Vol} W = (\operatorname{Vol} S^n)^{-1} \int_V dv \int_{B_v(1)} \operatorname{dist}(v, v'(\tau))\, d\tau,$$

where S^n is the unit sphere of dimension $n = \dim W - 1$, $v' = v'(\tau)$ is the point provided by the above conditions (1) and (2), and the interior integral is taken over the unit ball $B_v(1) \subset T_v(V)$.

Theorem (*Michel* [59]). *Let two compact connected $(n+1)$-dimensional Riemannian manifold (W, g) and (W', g') span the same manifold V,*

$$V = \partial W = \partial W' \text{ and } \operatorname{dist}_{g'}|V = \operatorname{dist}_g|V.$$

Then the manifolds W and W' are isometric in the following three cases:

(i) $n + 1 = 2$, *and the manifold W admits an injective Riemannian immersion into the hyperbolic plane H^2.*

(ii) *The manifold W admits an injective Riemannian immersion into a convex subset of the sphere S^{n+1} of curvature $+1$.*

(iii) *The manifold W admits a (possibly non-injective) Riemannian immersion f into $\mathbf{R}^{n+1}$.*

Proof. In all three cases we have $\operatorname{Vol} W' = \operatorname{Vol} W$. The equality $\operatorname{dist}_{g'} | V = \operatorname{dist}_g | V$ implies the homeomorphism of the tangent bundles of W and W', and thus we obtain (i) by the above remark (d).

The same argument applies to case (ii), but using Berger's isoembolic inequality (see §0.3) instead of Katok's theorem. For $n = 2$ one could also use Pu's theorem (compare (e) above).

Finally, under the assumptions of (iii) we have an isometric immersion $W' \to \mathbf{R}^{n+1}$ which extends the immersion $f | V$ (see §2.1). Then it is not hard to see that W' is isometric to W.

Warning. Without the distance condition $\operatorname{dist}_{g'} | V = \operatorname{dist}_g | V$, the existence of a Riemannian immersion $f'\colon W' \to \mathbf{R}^{n+1}$ such that

$$f'|(V = \partial W') = f|(V = \partial W)$$

does not imply the isometry (or even a mere homeomorphism) between W' and W (see [26]).

Remark. Further rigidity results as well as a more detailed proof of the above theorem (under slightly different assumptions) can be found in [59].

5.5.C. Let γ be a simple closed two-sided curve in a surface V, and let for some $\varepsilon > 0$, every noncontractible simple closed curve in the ε-neighborhood $U_\varepsilon(\gamma) \subset V$ be homotopic to γ.

Lemma. *The conformal length of the homotopy class* $[\gamma]$ *satisfies*

$$L[\gamma] = \text{conf length}\,[\gamma] \leqslant \tfrac{1}{2}\varepsilon^{-1}\sqrt{\text{Area}\, U_\varepsilon(\gamma)} \leqslant \tfrac{1}{2}\varepsilon^{-1}\sqrt{\text{Area}\, V}\,.$$

Proof. Use the measure $d\mu = d\alpha$ on the family of equidistant curves $\{\operatorname{dist}(v, \gamma) = \alpha\}$ for $0 \leqslant \alpha \leqslant \varepsilon$.

Examples. (a) If γ is a homologically systolic closed geodesic, then the ε-neighborhoods $U_\varepsilon(\gamma)$ satisfy the above condition for $0 \leqslant \varepsilon < \frac{1}{4}$ length γ.

(b) Furthermore, let V have constant curvature -1. Then

$$\text{Area}\, U_\varepsilon = (e^\varepsilon + e^{-\varepsilon})\ \text{length}\,\gamma,$$

and so

$$L[\gamma] < 6\ \text{length}\,\gamma, \quad \text{for length}\,\gamma \geqslant 1. \tag{5.9}$$

On the other hand, if length $\gamma \leqslant 1$, then the ε-neighborhood for $\varepsilon = \frac{1}{4}$ has the required property by the Kazdan-Margulis-Zassenhaus theorem (see [65], [66]), and *so the inequality* (5.9) *holds for an arbitrary homologically systolic closed geodesic in a complete surface of curvature* -1.

We obtain as a corollary the following improvement of the Blatter theorem (see §0.3) for surfaces of large genus $\to \infty$.

5.5.C′. Theorem. *Let V be a closed surface of genus g. Then there exists a closed curve γ in V, which is not homologous to zero and whose free homotopy class has*

$$\text{conf length}\,[\gamma] \leqslant \text{const}\log g,$$

for some constant $0 < \text{const} < 1000$.

Proof. The homologically systolic geodesic γ in a surface of constant curvature -1 has length $\lesssim$ log Area, since the ball $B_v(R)$ for every $v \in \gamma$ has

$$\text{area} \approx \exp R, \quad \text{for large } R \leqslant \tfrac{1}{2}\text{length}\,\gamma.$$

As any metric on V is conformal to a metric of constant curvature, the above example applies.

5.6. Isosystolic surfaces. A surface V is said to be *isosystolic* if $\text{sys}(V, v) = \text{sys}(V)$ for all $v \in V$.

5.6.A. Example. Let us fix the topological type of V, and consider the following functional sys Ar on the Riemannian metrics g on V:

$$\text{sys Ar}(g) = \text{Area}(V, g)/\text{sys}(V, g)^2.$$

Then the extremal metrics g, for which the functional sys Ar(g) assume the minimum, are clearly *isosystolic.*

The extremal metrics on $\mathbf{R}P^2$ have constant curvature by Pu's theorem. Flat hexagonal tori are also extremal by Loewner's theorem.

One could modify the definition of the extremal metric by considering Finsler metrics in addition to Riemannian metrics on V. This may lead to different extremal metrics as Example 5.2.B$''$ shows. However, we discuss below only Riemannian metrics on V.

5.6.B. Singular metrics. It is unlikely that there are nonsingular isosystolic (in particular extremal) metrics on surfaces V for $\chi(V) < 0$. However, there are interesting "singular Riemannian metrics" which are closely related to the classical extremal metrics of Grötzsch and Teichmüller (see [47]). These metrics have only finitely many singular points, and are flat outside these points. The metric near each singularity is isometric to a cone.

5.6.B$'$. Examples. Let V be the sphere S^2 with three holes (a pair of pants). This V can be decomposed into the union of three cylinders $S^1 \times [0, 1]$; see Fig. 1 below.

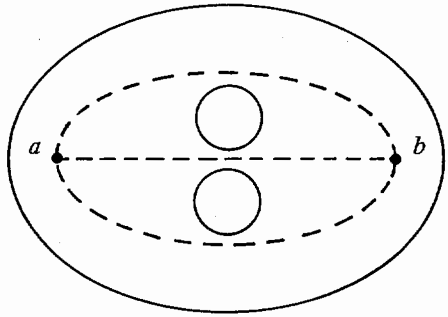

FIG. 1

Any two cylinders meet at an edge. These edges form a *triplet*, which consists of two vertices a and b joined by three edges.

Now let us equip the cylinders with the product metric of $S^1(2R) \times [0, l]$ where $2R =$ the length of the circle, and let the three circles $S^1(2R) \times 0$ meet across edges of length R. Then we get a singular metric on V with two singular points a and b. The boundary of this V consists of the three circles $S^1(2R) = S^1(2R) \times l$. Take the double of V. The resulting closed manifold V' has genus 2. The metric in V' is flat outside the four vertices. If $R \leqslant 2l$, then this metric is isosystolic with sys $V = 2R$.

Now let Γ be an arbitrary graph with q edges $e_1, \cdots, e_q$. We assign length $2l_i$ to each edge e_i, thus making $e_i = [0, 2l_i]$. Suppose that at every vertex of Γ there are exactly three edges of Γ. Then Γ has $p = \frac{2}{3}q$ vertices v_j, $j = 1, \cdots, p$. Next we take q cylinders $S^1(2R) \times [0, 2l_i]$, and identify triples of the boundary circles, according to the above (triplet) pattern, as these circles meet at some vertex v_j of Γ. Thus we get a surface V of the Euler characteristic $\chi(V) = -p$ with $2p$ singular points, two points over each vertex of Γ. There is a natural map P: $V \to \Gamma$, whose restriction to every cylinder $S^1(2R) \times [0, 2l_i]$ coincides with the projection onto the edge $[0, 2l_i]$. This map P is obviously distance-decreasing. Therefore if $2R \leqslant \text{sys}\, \Gamma < \Sigma_i 2l_i$, then the surface V is isosystolic.

It is unclear if every compact orientable surface admits an extremal metric which is almost everywhere flat as in this example. However, we shall prove below the existence of some *generalized extremal metric* on every compact surface V.

5.6.C. Generalized Riemannian metrics. A metric g on V is called a *generalized Riemannian metric* if the following two conditions are satisfied:

(1) g is a *length* metric, that is, the distance between any two points v_1 and v_2 in V equals the length of the shortest path between v_1 and v_2.

(2) There exists a surface of constant curvature, (V_0, g_0), and a homotopy equivalence f: $V_0 \to V$, which is "conformal" in the following sense. Denote by dist* the pull-back under f of the distance function dist_g on V. Then the function dist* on V_0 is the limit of a uniformly convergent sequence of metrics, dist_i on V_0, where each metric dist_i is given by a Riemannian metric on V_0, which is conformal to g_0, that is, $\text{dist}_i = \text{dist}_{g_i}$ for $g_i = \varphi_i^2 g_0$. We further require the sequence φ_i to converge in the L^2-norm to an L^2-function φ on V. We define Area$(V, g) =$ Area(V, g, f, φ) by

$$\text{Area}(V, g) = \int_{V_0} \varphi^2 = \lim_{i \to \infty} \text{Area}(V_0, g_i).$$

5.6.C′. Theorem. *An arbitrary closed surface V admits a generalized extremal metric g, at which the functional* sys Ar(g) (*see Example* 5.6.A) *assumes the minimum, denoted* Min Ar(V).

Proof. First let g_i be an arbitrary minimizing sequence of Riemannian metrics on V, that is, sys Ar$(g_i) \to$ Min Ar(V) for $i \to \infty$. Let us modify this sequence g_i by making it converge to a generalized metric, but let us keep the notation g_i unchanged.

We normalize the metrics g_i (as well as the metrics we shall have later) by the condition systole $= 1$. For such metrics Area $g =$ sys Ar g, assign to each metric g_i its conformal type, which is represented by a (normalized) metric g_i^0 of constant curvature. The results of §5.1 show the areas of g_i^0 to be uniformly bounded for $i \to \infty$. It follows (see [19]) that the sequence of metrics g_i^0 is precompact (in the modular space of conformal structures), and so we may assume (by passing to a subsequence if necessary) that the surfaces (V, g_i^0) converge to a surface (V_0, g_0) of constant curvature. This means that there is a sequence of metrics $\tilde{g}_i^0$ on V_0, which C^∞-converges to g_0 such that each surface $(V_0, \tilde{g}_i^0)$ is isometric to (V, g_i^0) for all $i = 1, \cdots$.

Now we replace the sequence of metrics g_i, which are isometric to $\varphi_i^2 \tilde{g}_i^0$, by the sequence $\varphi_i^2 g_0$, thus making all metrics of the minimizing sequence conformally equivalent to a fixed metric g_0 on V_0.

Next we apply to the sequence $g_i = \varphi_i^2 g_0$ the following.

5.6.C″. Regularization lemma. *Let (V, g) be an arbitrary closed surface with $\pi_1(V) \neq 0$. Then for an arbitrary $\varepsilon > 0$ there exists a conformal metric $\bar{g} = \bar{\psi}^2 g$ on V such that the following hold*:

(a) *The metrics $\bar{g}$ and g have a common set of minimal geodesics, and these geodesics have* $\text{length}_{\bar{g}} = \text{length}_g$. *In particular*, $\text{sys}(V, \bar{g}) = \text{sys}(V, g)$.

(b) *The function $\bar{\psi}$ satisfies*

$$0 < \bar{\psi}(v) \leqslant 1, \quad \text{for all } v \in V.$$

In particular, $\text{Area}(V, \bar{g}) \leqslant \text{Area}(V, g)$.

(c) *The metric $\bar{g}$ is ε-regular in the sense that its height function satisfies $h_{\bar{g}'}(v) \leqslant \varepsilon$ for all $v \in V$.*

Proof. Take a point $v \in V$ where $h(v) = h_g(v) > \varepsilon$, and take a function ψ, $0 < \psi \leqslant 1$, which is very small positive on the ball $B_v(\varepsilon/3)$ and is identically one outside the ball $B_v(\varepsilon/2)$. The metric $g' = \psi^2 g$ clearly satisfies (a) and (b). If g' is not ε-regular, we apply the above operation to g', and keep on doing this until we arrive (necessarily in finitely many steps) at an ε-regular metric.

Observe that the ε-regularity property (c) implies the following:

(c′) *Every $\bar{g}$-ball $B_v(R)$, for $\varepsilon \leqslant R \leqslant \text{sys}(V, \bar{g})$, has*

$$\text{Area}_{\bar{g}} B_v(R) \geqslant \tfrac{1}{2} R^2.$$

(c″) *Every closed simple curve in $(V, \bar{g})$ of diameter $d < \frac{1}{2}\text{sys}(V, \bar{g})$ bounds a disk in $(V, \bar{g})$ of diameter $\leqslant 3(d + \varepsilon)$.*

Proof. (c′) follows from Proposition 5.1.B whose *proof* yields (c'') as well.

Now we may assume the metrics $g_i = \varphi_i^2 g_0$ on V_0 to be ε_i-regular for $\varepsilon_i \to 0$ as $i \to \infty$. The proprty (c′) makes such a sequence of metric spaces (V_0, g_i) *precompact* in the abstract Hausdorff topology (see Appendix 3), and so we may assume (passing to a subsequence if necessary) this sequence to converge to a compact metric space $V_\infty = (V_\infty, g_\infty)$.

Next we apply the theory of *k-regular convergence* (see [76]) and use (c″) above to conclude that the subspace $V_\infty \subset V$ is, in fact, homeomorphic to V. The limit metric in this $V \approx V_\infty$ is indeed a length metric (see Appendix 3). Furthermore, the sequence of functions $\varphi_i^2 = g_i/g_0$ does L^2-converge to some φ^2. This is an obvious consequence of the convexity of the L^2-norm (see [48]). Let us show that the sequence of maps f_i is uniformly continuous relative to the fixed metric g_0 on V_0. Take two points v_1 and v_2 in some $g_0 =$ -all $B(\rho) \subset B(R)$ for a fixed positive $R < \frac{1}{2}\mathrm{sys}(V, g_0)$ and a small $\rho < R$. If $\rho/R \to 0$, then the conformal length of the family Γ of the concentric circles $S(r)$ in the annulus $B(R) \setminus B(\rho)$ also goes to zero. Indeed, with the measure $d\mu =$ length $S(r)\, dr$ we get (compare §5.5) conf length $\Gamma \leqslant \text{const}\,|\log \rho\,|^{-1}$.

The property (c″) now implies that

$$\mathrm{dist}(f_i(v_i), fd(v_2)) = \mathrm{dist}_{g_i}(v_1, v_2) \leqslant \mathrm{const}\left|\log \mathrm{dist}_{g_0}(v_1, v_2)\right|^{-1},$$

for all pairs of points v_1 and v_2 in V_0 and for some constant const > 0, which depends only on the topology of V_0.

The uniform continuity of the map f_i allows us to find a subsequence which converges to a map $f\colon V_0 \to V \approx V_\infty \subset X$, which clearly is the required homotopy equivalence.

Remark. It is not hard to see that the limit metric g_∞ on $V \approx V_\infty$ is uniquely determined by the choice of the metric g_0. However, it is unclear whether this g_0 is unique, and whether the manifold (V_0, g_0) has a nontrivial group of isometries.

6. Minimal geometric cycles

Take an arbitrary discrete group Π, and let h be an n-dimensional homology class of Π: $h \in H_n(\Pi) \overset{\mathrm{def}}{=} H_n(K(\Pi, 1))$, where the homology is understood with coefficients $\mathbf{Z}$ or $\mathbf{Z}_2$. A *geometric cycle, which represents h*, by definition is an n-dimensional pseudomanifold V with a piecewise smooth Riemannian metric and a map $f\colon V \to K(\Pi; 1)$ such that $f_*[V] = h$. If h is an *integral* class, then $[V]$ is an oriented fundamental class of V. If h is a $\mathbf{Z}_2$-class, then V need not be orientable. In either case we do not assume V to be connected.

We denote by $\operatorname{sys}(V, v) = \operatorname{sys}(V, v; f)$ the length of the shortest loop γ in V with the base point $v \in V$, whose image under f is not contractible. We put

$$\begin{aligned}\operatorname{sys}(V) &= \operatorname{sys}(V; f) = \inf_{v \in V} \operatorname{sys}(v, v),\\ \operatorname{sys}\operatorname{Vol}(V) &= \operatorname{Vol} V/(\operatorname{sys} V)^n,\\ \operatorname{sys}\operatorname{Vol} h &= \inf_V \operatorname{sys}\operatorname{Vol} V,\end{aligned}$$

over all geometric cycles V which represent h.

We denote by $\tilde{K} \to K = K(\Pi; 1)$, the universal covering of our space $K(\Pi; 1)$, and then we have the covering Π-equivariant map $\tilde{f}$: $\tilde{V} \to \tilde{K}$. The systole of V can be expressed in terms of the *displacements* of the deck isometries π: $\tilde{V} \to \tilde{V}$, $\pi \in \Pi$, as follows:

$$\operatorname{sys}(V, f) = \inf_{\pi, \tilde{v}} \operatorname{dist}(\tilde{v}, \pi(\tilde{v})),$$

over all $\pi \neq \mathrm{id}$ in Π and all $\tilde{v} \in \tilde{V}$.

Our estimate 1.3 applies to all geometric cycles V and shows that

$$\operatorname{sys}\operatorname{Vol} h \geqslant \mathrm{const}_n > 0.$$

Our aim is to sharpen this estimate under some additional assumptions on the class h.

6.1. A geometric model for the $K(\mathbf{Z}_2; 1)$-space. We want to construct a universal metric on the space $K = K(\mathbf{Z}_2; 1)$ for which the *volume* of every homology class $h \in H_n(K; \mathbf{Z}_2) \approx \mathbf{Z}_2$ equals the systolic volume of h. Recall that $\operatorname{Vol} h$ by definition is the lower bound of the volumes of cycles which represent h.

Take the L^∞-space of bounded Borel functions on an infinite compact connected metric space, and let $\tilde{K}$ be the sphere of radius $\frac{1}{2}$ in this L^0:

$$\tilde{K} = \left\{x \mid \|x\|_{L_\infty} = \tfrac{1}{2}\right\}.$$

We equip $\tilde{K}$ with the induced *length-metric* $\operatorname{dist}(x_1, x_2) = \inf$(the lengths of curves in $\tilde{K}$ between x_1 and x_2), and then divide $\tilde{K}$ by the involution $x \to \infty x$.

6.1.A. Proposition. *The quotient space* $K = \tilde{K}/\{-1, +1\}$ *with the induced metric is the required universal space.*

Proof. The space $\tilde{K}$ is obviously aspherical, and so K is a $K(\mathbf{Z}_2; 1)$-space. It is clear that $\operatorname{sys}_1(K) \geqslant 1$, and so $\operatorname{sys}\operatorname{Vol} h \leqslant \operatorname{Vol} h$. To prove the inequality $\operatorname{Vol} h$ we must construct, for every geometric cycle (V, f) with $\operatorname{sys}(V, f) \geqslant 1$, a distance nonincreasing $\mathbf{Z}_2$-equivariant map $\tilde{I}$: $\tilde{V} \to \tilde{K}$ for the canonical double covering $\tilde{V}$ of V. Let

$$\delta_{\tilde{v}}(\tilde{w}) = \min[\operatorname{dist}(\tilde{v}, \tilde{w}), 1],$$

for all $\tilde{v}$ and $\tilde{w}$ in $\tilde{V}$. Then we take for $\tilde{I}(\tilde{v})$ the following function $\varphi_{\tilde{v}}\colon \tilde{V} \to \mathbf{R}$:

$$\varphi_{\tilde{v}}(w) = \tfrac{1}{2}\left[\delta_{\tilde{v}}(\tilde{w}) - \delta_{\tilde{v}}(\pi(\tilde{w}))\right],$$

for the deck involution π on $\tilde{V}$. The function $\varphi_{\tilde{v}} \in L^\infty = L^\infty(\tilde{V})$ is contained in $\tilde{K} \subset L^\infty$ for every $\tilde{v} \in \tilde{V}$, and the map $\tilde{I}\colon \tilde{V} \to \tilde{K}$ clearly is locally isometric.

Remark. The above construction shows that $\operatorname{sys}_1(K) = 1$, as the map $\tilde{I}$: $S^1 \to L^\infty(S^1)$ sends the circle S^1 of length 2 isometrically onto a centrally symmetric curve (of length 2) in the sphere $S^\infty(\frac{1}{2}) = \tilde{K} \subset L^\infty = L^\infty(S^1)$. Observe that this map $\tilde{I}$ (and the image curve as well) is nowhere differentiable.

In fact, if a *closed* curve γ in an arbitrary Banach space L is differentiable at some point $x_0 \in \gamma$, then

$$2\operatorname{dist}(x, x_0) < \operatorname{length}\gamma,$$

for all points $x \in \gamma$. Indeed, take the linear (or rather affine) projection l of norm one of L onto the line through the points x and x_0. If $\operatorname{dist}(x, x_0) = \frac{1}{2}$ length γ, then length $l(\gamma) =$ length γ. If γ were differentiable at x_0, then the derivative of the projection $l \mid \gamma\colon \gamma \to \mathbf{R}$ would be zero at x_0, which would make length $l(\gamma) <$ length γ. q.e.d.

It follows that every centrally symmetric curve γ in the $\frac{1}{2}$-sphere of *a finite dimensional* Banach space L has length $\gamma > 2 + \varepsilon_n$, for $\varepsilon_n > 0$ and $n = \dim L$. (Compare [69].)

6.2. Cubical complexes. The standard δ-*cube* in a (finite or infinite dimensional) space $L^\infty = L^\infty(X)$ by definition is the set of function $\{\varphi(x) \mid 0 \leqslant \varphi(x) \leqslant \delta$, for all $x \in X\}$. A cubical δ-complex is then defined as a metric space K which is partitoned into (isometric images of) δ-cubes such that any two cubes meet at a face.

6.2.A. Example. The hyperplanes $\{\varphi(x) = m\delta\}$ in $L^\infty(X)$ for $x \in X$ and $m = \cdots, -1, 0, +1, \cdots$, partition of the space $L^\infty(X)$ into δ-cubes.

We agree, whatever the original metric in K was, to use the associated *length metric* in K, which is the greatest metric in K, compatible with the L^∞-metrics in the cubes.

Let K be an arbitrary δ-complex, and ε a number in the interval $0 \leqslant \varepsilon < \frac{1}{2}\delta$. Let us construct a map $R_\varepsilon\colon K \to K$ with the following two properties:

(1) R_ε is a piecewise linear Lipschitz map with the Lispchitz constant $(\frac{1}{2}\delta - \varepsilon)^{-1}$.

(2) The map R_ε retracts the ε-neighborhood $U_\varepsilon(K_0)$ of an arbitrary subcomplex $K_0 \subset K$ onto K_0.

The required map obviously exists (and unique) for K equal to the interval $[0, \delta]$. Using this map which is called $r_\varepsilon\colon [0, \delta] \to [0, \delta]$, we have the map R_ε of the standard cube, $\varphi(x) \to r_\varepsilon \circ \varphi(x)$, and so we get the required map of K by applying this R_ε to every δ-cube of K.

6.2.B. An isoperimetric inequality for δ-complexes. *For a given integer $n = 1, \cdots$, there exist two positive constants μ_n and C'_n such that every singular n-dimensional cycle z of volume $\leqslant \mu_n \delta^n$ in an arbitrary δ-complex K bounds a chain c in K of* $\mathrm{Vol}(c) \leqslant C'_n(\mathrm{Vol}\, z)^{n+1/n}$, *such that c is contained in the ε'-neighborhood of (the support of) z for $\varepsilon' = C'_n(\mathrm{Vol}\, z)^{1/n}$.*

Proof. First let K be a subcomplex of the above δ-subdivision of the space L^∞. (In fact, only this case is needed for our present purposes.) Then we can span the cycle $z \subset K \subset L^\infty$ by a chain $\tilde{c}$ in L^∞, which has

$$\mathrm{Vol}\, \tilde{c} = \mathrm{Fill\,Vol}\, z \leqslant C_n(\mathrm{Vol}\, z)^{n+1/n},$$

and is ε-close to K for $\varepsilon < \mathrm{const}_n(\mathrm{Vol}\, z)^{1/n}$ (see §4.2, 4.3). Thus we apply the map $R_\varepsilon \colon L^\infty \to L^\infty$, and take the sum

$$R_\varepsilon(\tilde{c}) + \mathrm{Cylinder}(R_\varepsilon | z),$$

for the required chain c.

Now for an arbitrary δ-complex K we observe the following obvious "cone inequality" (compare §4.1), for all k-dimensional cycles z in K of diameter $\leqslant \delta/3$:

$$\mathrm{Fill\,Vol}(z \subset K) \leqslant C(k)(\mathrm{Vol}\, z)(\mathrm{Diam}\, z).$$

With such an inequality the argument of §3.4 yields the isoperimetric inequality. (See Appendix 2 for a more general isoperimetric inequality.)

6.3. δ-Extensions of geometric cycles. Let Π be an arbitrary group, and let (V, f) for $f\colon V \to K = K(\Pi; 1)$ be an n-dimensional geometric cycle. Take a subset $V_0 \subset V$ and let $I_0 = I_0(V_0, \delta)\colon V \to L_0^\infty = L^\infty(V_0)$, for some $\delta > 0$, be the map

$$I_0 = v \to \varphi_v(w) = \min[\delta, \mathrm{dist}(v, w)],$$

for all $v \in V$ and $w \in V_0$.

Observe that for $V = V_0$ the map I_0 is locally isometric. Furthermore, if V_0 is "sufficiently" dense in V, then I_0 is "almost" locally isometric, where "almost" depends on how dense V_0 is in V.

Next we choose a small positive number $\varepsilon < \frac{\delta}{2}$, and apply the map R_ε of the previous section to the standard δ-subdivision of the space L_0^∞.

We assign to every point $x \in L_0^\infty$ the unique minimal δ-cube $\square_x \subset L_0^\infty$ of the canonical δ-subdivision of L_0^∞, which contains x, and to each point $v \in V$ the cube $\square(v) = \square_x$ for $x = R_\varepsilon \circ I_0(v)$. We call *a δ-extension of V* the union

$$K_\delta(V) = K_\delta(V, V_0, \varepsilon) = \bigcup_{v \in V} \square(v),$$

and denote by $J = J(V, V_0, \varepsilon, \delta)$ the composition map $J = R_\varepsilon \circ I_0\colon V \to K_\delta(V) \subset L_0^\infty$.

Next we turn to the covering map $\tilde{f}\colon \tilde{V} \to \tilde{K}$ for the universal covering $\tilde{K} \to K(\Pi; 1)$. We denote by $\tilde{V}_0 \subset \tilde{V}$ the lift of V_0 to $\tilde{V}$, and assume that $\operatorname{sys}(V, f) \geqslant 2\delta$. Then we lift the map I_0 to the following Π-equivariant map $\tilde{I}_0$: $\tilde{V} \to \tilde{L}_0^\infty = L^\infty(\tilde{V}_0)$:

$$\tilde{I}_0\colon \tilde{v} \to \min[\delta, \operatorname{dist}(\tilde{v}, \tilde{w})].$$

We observe that the map $R_\varepsilon\colon L_0^\infty \to L_0^\infty$ lifts to the corresponding R_ε-map of $\tilde{L}_0$, called $\tilde{R}_\varepsilon\colon \tilde{L}_0 \hookleftarrow$. The complex $K_\delta(V)$ lifts to a Π-invariant subcomplex $\tilde{K}_\delta(V) = K_\delta(\tilde{V})$ of the δ-subdivision of $\tilde{L}_0^\infty$, and the map J lifts to a map $\tilde{J}$: $\tilde{V} \to \tilde{K}_\delta(V)$.

Notice that the map $\tilde{J}$, as well as J, is Lipschitz with the Lipschitz constant $\leqslant \delta/\delta - 2\varepsilon$.

6.3.A. Lemma. *Let the subset V_0 be ε'-dense in V for some nonnegative $\varepsilon' \leqslant \varepsilon/10$. Let $\tilde{v}$ and $\tilde{v}'$ for a pair of points in $\tilde{V}$, and let $\square$ and $\square'$ be some δ-cubes in $\tilde{K}_\delta(V)$, which contain the images $\tilde{J}(\tilde{v})$ and $\tilde{J}(\tilde{v}')$ respectively. If* $\operatorname{dist}(\tilde{v}, \tilde{v}') \geqslant m\delta$ *for some integer $m = 0, 1, \cdots$, then also* $\operatorname{dist}(\square, \square') \geqslant m\delta$.

Proof. If $\operatorname{dist}(\tilde{v}, \tilde{v}') \geqslant \delta$, then the cubes $\square$ and $\square'$ do not intersect, and so the distance between them (i.e., the length of the shortest path in $\tilde{K}_\delta(V)$ between $\square$ and $\square'$) is at least δ.

Now in the general case we take the shortest (or an almost shortest) path between $\square$ and $\square'$, and take a point $\tilde{w}$ on this path, for which $\operatorname{dist}(\square, \tilde{w}) = \delta$. This point is contained in some cube $\square'' = \square''(\tilde{v}'')$ for some point $\tilde{v}''$ in $\tilde{V}$ such that $\tilde{J}(\tilde{v}'') \in \square''$. As

$$\operatorname{dist}(\square, \square') \geqslant \delta + \operatorname{dist}(\square'', \square'),$$

and

$$\operatorname{dist}(\tilde{v}'', \tilde{v}') \geqslant \operatorname{dist}(\tilde{v}, \tilde{v}') - \operatorname{dist}(\tilde{v}, \tilde{v}'') \geqslant \operatorname{dist}(\tilde{v}, \tilde{v}') - \delta,$$

the proof follows by induction on m.

If the numbers ε and ε' are sufficiently small, then we may assume without loss of generality that the map $J\colon V \to K_\delta(V)$ is an embedding on a sub-pseudomanifold $V' = J(V) \subset K_\delta(V)$. Thus the pair $(V', f' = f \circ J^{-1})$ is also a geometric cycle which is, in the obvious sense, homotopic to (in fact, close to) (V, f). The cycle V' clearly has

$$\operatorname{Vol} V' \leqslant (\delta/\delta - 2\varepsilon)^n \operatorname{Vol} V.$$

Furthermore, the map $f'\colon V' \to K(\Pi, 1)$ extends to $K_\delta(V) \supset V'$, since the covering map $\tilde{f}'\colon \tilde{V}' \to \tilde{K}$ equivariantly extends to $\tilde{K}_\delta(V) \supset \tilde{V}'$. Thus every subpseudomanifold (cycle) V'' in $K_\delta(V)$, which is homologous to V', realizes the class $h = f_*[V] \in H_n(K(\Pi; 1))$.

Now the above lemma implies the following.

6.3.B. Corollary. *If the systole of V is an integral multiple of δ, $\mathrm{sys}(V) = m\delta$, then $\mathrm{sys}(V'') \geqslant \mathrm{sys}\, V$. Moreover, if γ is the shortest curve in V, whose image in $K(\Pi, 1)$ represents a given free homotopy class of curves, then the corresponding shortest curve γ'' in V'' has*

$$\text{length}\, \gamma'' \geqslant \delta\, \text{ent}\big(\delta^{-1}\, \text{length}\, \gamma\big),$$

where "ent" *denotes the entire part of the number δ^{-1} length γ.*

6.3.C. Remark. There is a natural cubical model K of any $K(\Pi, 1)$ space such that

$$\mathrm{sys\,Vol}\, h = \mathrm{Vol}\, h,$$

for an arbitrary homology class $h \in H_n(\Pi)$. Namely, let the group Π naturally act on the space $X = \Pi \times [0, 1]$. Take the space $\tilde{K}$ of those Borel functions φ: $X \to [0, 1]$, for which

$$\|\varphi - \varphi \circ \pi\|_{L_\infty} = 1,$$

for every element $\pi \neq \mathrm{id}$ in Π. This space $\tilde{K}$ has a natural structure of a cubical complex (for $\delta = 1$), and it is not hard to see that K is aspherical, provided the group Π is countable. Furthermore, the covering $\tilde{V}$ of every geometric cycle admits a locally isometric equivariant map into this $\tilde{K}$ as long as $\mathrm{sys}\, \tilde{V} \geqslant 1$. Therefore the space $K = \tilde{K}/\Pi$ is our model.

6.4. Regulation of geometric cycles. A geometric cycle $V = (V, f)$ is said to be *ε-regular*, if every ball $B_v(R)$ in V, for all $v \in V$ and $\varepsilon \leqslant R < \mathrm{sys}\, V$, has

$$\mathrm{Vol}\, B_v(R) \leqslant (1 + \varepsilon)\mathrm{Fill\,Vol}\, \partial B_v(R). \tag{6.1}$$

6.4.A. Theorem. *Every homology class $h \in H_n(\Pi)$ for an arbitrary group Π can be represented by an ε-regular cycle V such that*

$$\mathrm{sys\,Vol}\, V \leqslant (1 + \varepsilon)\mathrm{sys\,Vol}\, h, \tag{6.2}$$

where ε is an arbitrarily small positive number.

Proof. We start with a cycle V_1 for which $\mathrm{sys\,Vol}\, V_1 \leqslant (1 + \varepsilon_1)\mathrm{sys\,Vol}\, h$ where ε_1 is a positive number much smaller than ε. It is convenient to normalize V_1 by the condition $\mathrm{sys}\, V_1 = 1$, which makes $\mathrm{Vol}\, V_1 = \mathrm{sys\,Vol}\, V_1$. If inequality (6.1) is violated for some ball $B_v(R)$ in V_1, then the volume of this ball must be quite small:

$$\mathrm{Vol}\, B_v(R)/\mathrm{Vol}(V_1) \leqslant 2\varepsilon_1/\varepsilon.$$

To exclude these 'bad" balls, we take, with a finite very dense subset $V_0 \subset V_1$, a δ-extension $K_\delta(V_1)$, which is a compact polyhedron, since V_0 is finite. Then we take a (minimizing) sequence of sub-pseudomanifolds (cycles) V_i' in $K_\delta(V_1)$, which are homologous to $V_1' \subset K_\delta(V_1)$, such that the volumes $\operatorname{Vol}(V_i')$ converge to the volume of the homology class $[V_1'] \in H_n(K_\delta(V_1))$. As the set V_0 is very dense in V_1, we may assume

$$\operatorname{Vol} V_i' \leqslant (1 + \varepsilon/2) \operatorname{sys} \operatorname{Vol} h, \quad \text{for all } i = 1, \cdots .$$

As the polyhedron $K_\delta(V_1)$ is compact, we may assume (by taking a subsequence if necessary) that the sequence V_i' Hausdorff converges to a compact subset V'' in $K_\delta(V_1)$. We may further assume this V'' to be *minimal*: no proper subset of V'' is the limit of any other minimizing sequence. (Compare §4.3.C.)

As the space $k_\delta(V_1)$ satisfies the isoperimetric inequality 6.2.B, the balls $B_v(R)$ in V'' have

$$\operatorname{Vol} B_v(R) \geqslant A_{n-1}' R^n,$$

where the volume by definition is a weak limit of volumes of the cycles V_i'. Compare Theorem 4.3.C″, and so the approximating cycles V_i' enjoy the same inequality for all balls of radius $\geqslant \varepsilon = \varepsilon(i) \to 0$ as $i \to \infty$.

6.4.B. Corollary. *The above "regular representatives" V of h have the balls $B_v(R)$ such that*

$$\operatorname{Vol} B_v(R) \geqslant (1 - \varepsilon) A_{n-1} R^n, \tag{6.3}$$

for all $v \in V$, $\varepsilon \leqslant R \leqslant \frac{1}{2}\operatorname{sys}(V, v)$, the constant A_{n-1} of Theorem 4.3.C″, and an arbitrarily small $\varepsilon < 0$.

Indeed the proof of Theorem 4.3.C″ applies.

6.4.B′. Remark. The estimate (6.3) shows that there is a minimizing sequence of cycles V_i,

$$\operatorname{sys} V_i = 1, \quad \operatorname{Vol} V_i \to \operatorname{sys} \operatorname{Vol} h,$$

which converges in the abstract Hausdorff topology to a generalized minimal cycle V^* (compare §5.6). This V^* can be isometrically imbedded into the cubical model K (see Remark 6.3.C) of our $K(\Pi; 1)$ space. The inequality (6.3) holds for all balls $B_v(R)$ in V^* for $R \in [0, 1 = \operatorname{sys} V^*]$. This amounts to the same inequality (6.3) for approximating ε-regular cycles $V = V(\varepsilon)$ and for $\varepsilon \to 0$. To simplify notation, we allow ourselves to treat this V^* as if it were a regular (i.e., ε-regular for $\varepsilon = 0$) geometric cycle.

6.4.C. Estimates for Betti numbers. Cover a geometric cycle $V = (V, f)$ by balls. Let P denote the nerve of this cover, and let p: $V \to P$ be the map associated to a partition of unity attached to this cover. If the balls have radii $< \frac{1}{8}\operatorname{sys} V$, then there is a map $g\colon P \to K(\Pi, 1)$ such that the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & K(\Pi, 1) \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle g} & \\ P & & \end{array}$$

(Compare §1.2 and Theorem 5.3.B.) In particular, the homology class of V, that is, $p_*[V] \in H_n(P)$, is sent by the map g to the class $h = f_*[V]$ in the group $H_n(K(\Pi; 1))$.

Now let V^* be a regular cycle. By taking a maximal system of α-admissible balls in V^* we obtain, as in the proof of Theorem 5.3.B, an "efficient" cover of V^* by balls of radii $< \frac{1}{8}\operatorname{sys} V$ such that the total number of k-simplices in the nerve P of this cover satisfies

$$N_k = N_k(P) \leqslant s \exp C\sqrt{\log s}\,, \quad k = 0, 1, \cdots, n, \tag{6.4}$$

for $s = \operatorname{sys} \operatorname{Vol} h$ and some universal positive constant $C = C(n)$.

Summarizing the above discussion gives the following.

6.4.C′. Theorem. *For an arbitrary homology class $h \in H_n(K(\Pi, 1))$, there exists a polyhedron P satisfying inequalities* (6.4) *and a map $g\colon P \to K(\Pi, 1)$ sending some class $h' \in H_n(P)$ to h.*

By this theorem one can relate some algebraic invariants of h to the systolic volume of h.

Let us introduce the following ranks $\operatorname{rank}_k(h; F)$, for $k = 1, \cdots, n-1$, where F is an arbitrary coefficient field of h is an integral class, and $F = \mathbf{Z}_2$ if h is a mod 2 homology class. We evaluate the cohomological cup product:

$$H^k(K(\Pi; 1); F) \cup H^{n-k}(K(\Pi; 1); F),$$

on the class h, and we take the rank of the resulting bilinear form for our $\operatorname{rank}_k h = \operatorname{rank}_k(h; F)$.

6.4.C″. Theorem. *The above ranks satisfy*

$$\operatorname{rank}_k h \leqslant (\operatorname{sys} \operatorname{Vol} h) \exp\left[C\sqrt{\log(\operatorname{sys} \operatorname{Vol} h)}\,\right],$$

for all $k = 0, \cdots, n$ and some universal constant $C = C(n)$.

Proof. The class h is the image of $h' \in H_n(P)$, $h = g_*(h')$, and so $\operatorname{rank}_k h \leqslant \operatorname{rank}_k h' \leqslant \operatorname{rank} H^k(P) \leqslant N_k(P)$.

6.4.C‴. Corollary. *The Betti numbers of an arbitrary closed n-dimensional aspherical manifold V satisfy, for all $k = 0, \cdots, n$,*

$$b_k(V) \leqslant s \exp C\sqrt{\log s}\,, \quad s = \operatorname{Vol} V / (\operatorname{sys}_1 V)^n.$$

Proof. The Poincaré duality implies $\operatorname{rank}_k[V] = b_k(V)$.

Remark. In the same way as above one can estimate the minimal number of the generators of the fundamental group $\pi_1(V)$.

6.4.D. An estimate for the simplicial norm of h. Let h be an n-dimensional homology class with real coefficients: $h \in H_n(K; \mathbf{R})$ for $K = K(\Pi, 1)$. Let $c = \sum_i r_i \sigma_i$ be singular cycles (with real coefficients r_i) which represent h, and let

$$\|h\| = \inf_c \sum_i |r_i|,$$

where c runs over all cycles homologous to h. The basic properties of this *simplicial norm* $\| h \|$ are discussed in [74] and [32]. We shall need the following two facts.

(a) **Smoothing inequality** (*see* [32, §2.4]). Let the class h be represented by a geometric cycle (V, f), $f\colon V \to K$. Let $\tilde{f}\colon \tilde{V} \to \tilde{K}$ be the covering map. Suppose we are given a nonnegative Π-invariant function $\mathbb{S}(y, y')$ of two variables y and y' in $\tilde{V}$, which has the following three properties:

(1) The function $\mathbb{S}$ has its support in some ε-neighborhood ($\varepsilon < \infty$) of the diagonal $\Delta \subset \tilde{V} \times \tilde{V}$.

(2) $\mathbb{S}$ is bounded and almost everywhere differentiable.

(3) $\mathbb{S}$ is symmetric, that is, $\mathbb{S}(y, y') = \mathbb{S}(y', y)$.

Put

$$\|D_y \mathbb{S}\| = \int_{\tilde{V}} \|\operatorname{grad}_{y'} \mathbb{S}(y, y')\| dy',$$

$$[\mathbb{S}]_y = \|D_y \mathbb{S}\| / \int_{\tilde{V}} \mathbb{S}(y, y')\, dy', \; [S] = \sup_{y \in \tilde{S}} [S]_y.$$

Then

$$\|h\| \leqslant \operatorname{const}_n [\mathbb{S}]^n \operatorname{Vol} V.$$

(b) **Thurston's inequality** (*see* [74], [32]). *If h is the fundamental class of a compact manifold V of negative curvature, i.e., $h = [V]$ for curvature $(V) \leqslant -k^2$, then*

$$\|h\| \geqslant \operatorname{const}'_n k^n \operatorname{Vol} V,$$

for some universal positive constant $\operatorname{const}'_n \geqslant 0$.

Now let V^* be a regular cycle which represents h, and let

$$\mathbb{S}_\lambda(y, y') = \begin{cases} \exp[-\lambda \operatorname{dist}(y, y')] - \exp(-\lambda R_0), & \text{for dist} \leqslant R_0, \\ 0, \quad \text{for dist} \geqslant R_0, \end{cases}$$

where λ is a positive number, and $R_0 = \operatorname{sys} V^*/2$. Then clearly

$$[\mathbb{S}]_y \leqslant \lambda I_\lambda/[I_\lambda - (\exp - \lambda R_0)\operatorname{Vol} V^*],$$

for

$$I_\lambda = \int_{\tilde{V}^*} \exp(-\lambda \operatorname{dist}(y, y'))\, dy'.$$

Inequality (6.3) implies

$$I_\lambda \geqslant A_{n-1}\left(\frac{R_0}{2}\right)^n \exp \frac{\lambda R_0}{2},$$

and so if

$$\lambda = \frac{2}{R_0} \log\left[\frac{2^{n+1} \operatorname{Vol} V^*}{A_{n-1} R_0^n}\right],$$

then $[\mathbb{S}]_y \leqslant 2\lambda$ for all $y \in \tilde{V}^*$. Finally we apply (a) above to conclude the following.

6.4.D′. Theorem. *The simplicial norm of h has the following upper bound in terms of the systolic volume of h:*

$$\|h\| \leqslant 4^n(\operatorname{sys} \operatorname{Vol} h)\log(B \operatorname{sys} \operatorname{Vol} h), \quad \text{for } B = 2^{n+1}/A_{n-1}.$$

Applying this theorem to the fundamental class h of a compact manifold V_0 of negative curvature $\leqslant -1$ and using (b) above we obtain the following corollary, which sharpens Theorem 0.2.

6.4.D″. Corollary. *Let V be a Riemannian manifold homeomorphic to V_0, and let $s = \operatorname{sys} \operatorname{Vol} V = \operatorname{Vol} V/(\operatorname{sys}_1 V)^n$. Then*

$$\operatorname{Vol} V_0 \leqslant C_n s \log(C_n' s),$$

for some universal positive constants C_n and C_n'. In particular, closed surfaces V of genus $g > 1$ have $C_2 s \log C_2' s \geqslant 4\pi(g-1)$. (Compare §5.3.)

6.5. Systems of short curves in aspherical manifolds. Let V be a closed oriented aspherical Riemannian manifold of dimension n. We want to locate as many as possible "independent" closed curves in V of relatively small length.

Let (V^*, f), $f\colon V^* \to V$, be a regular geometric cycle, which represents the fundamental class $h = [V] \in H_n(V)$ such that (see §§6.3, 6.4)

(a) $\operatorname{Vol} V^* \leqslant \operatorname{Vol} V$,

(b) $\operatorname{sys}(V^*, f) = \operatorname{sys}_1(V)$,

(c) $\operatorname{Vol} B_v(R) \geqslant A_{n-1} R^n$, for all balls in V^* of radius $R \leqslant \operatorname{sys}_1(V)$ and for the constant A_{n-1} of Theorem 4.3.C″.

By using a sufficiently small δ (see Corollary 6.3.B), we may further assume the following additional property of this V^*:

(d) The map $f\colon V^* \to V$ does not increase the lengths of the free homotopy classes of closed curves in V^*.

Take the shortest curve γ_1^* in V^*, whose image curve $f(\gamma_1^*)$ in V is not contractible, and let γ_1 denote the shortest geodesic in V homotopic to $f(\gamma_1^*)$. First we take the next shortest curve γ_2^* in V^* such that the (homotopy class of) corresponding geodesic γ_2 in V is not contained in the normal subgroup $N(\gamma_1)$ of $\pi_1(V)$ spanned by γ_1. Then we take the shortest curve γ_3^* for which γ_3 is not contained in the normal subgroup $N(\gamma_1, \gamma_2) \subset \pi_1(V)$, and so on. This process necessarily stops in finitely many steps, and finally we have some geodesics γ_i, $i = 1, \cdots, q$, which normally span the fundamental group

$$N(\gamma_1, \cdots, \gamma_q) = \pi_1(V).$$

In particular the (homology classes of) curves γ_i span the first homology group $H_1(V)$. By the above property (b) we can apply the arguments of §5.3, and then we get the following upper bounds for $l_i^* = \text{length}\,\gamma_i^*$ and thus for $l_i = \text{length}\,\gamma_i \leqslant l_i^*$.

6.5.A. Theorem. *The lengths l_i satisfy*

(1) $l_1 = \text{sys}_1 V \leqslant 2(\text{Vol}\,V/A_{n-1})^{1/n}$, *for the constant A_{n-1} of Theorem* 4.3.C″.

(2) $l_i \leqslant l_1 + 2^n \text{Vol}\,V/l_1^{n-1}A_{n-1}$, $i = 2, \cdots, q$, (*compare* §5.3.C).

(3) *The number q_l of those geodesics γ_i, which have length $\geqslant l$, is bounded by*

$$q_l \leqslant 10^n\left(\text{Vol}\,V/ll_1^{n-1}A_{n-1}\right)^2,$$

for every $l \geqslant 4l_1$. (*Compare Proposition* 5.3.C′.)

(4) *The total length of the geodesics γ_i is bounded by*

$$\sum_{i=1}^{q} l_i \leqslant (200)^n l_1^{1-3n}(\text{Vol}\,V/A_n)^3$$

(*Compare Proposition* 5.3.*D*.)

Next we generalize Theorem 5.3.E as follows. Let k be the greatest integer such that for arbitrary elements $\alpha_1, \cdots, \alpha_k$ in $\pi_1(V)$ the fundamental class $h = [V]$ does not vanish under the quotient homomorphism of groups: Q: $\pi_1(V) \to \Pi = \pi_1(V) \to \Pi = \pi_1(V)/N(\alpha_1, \cdots, \alpha_k)$.

6.5.A′. Theorem. *The curves γ_i, for $i = 1, \cdots, k+1$, have lengths*

$$l_i \leqslant 2\left(\text{Vol}\,V/A_{n-1}\right)^{1/n}.$$

Proof. Apply (1) above to the (nonzero!) class $Q_*(h) \in H_n(K(\Pi; 1))$.

Some open questions. Let a geometric cycle (V, f) represent the fundamental class of the n-torus T^n. Then one expects that there are some closed curves $\gamma_1, \cdots, \gamma_n$ in V, whose image curves $f(\gamma_i)$ in T^n generate the group $H_1(T; \mathbf{R}) \approx \mathbf{R}^n$, such that the lengths l_i of γ_i satisfy

$$\prod_{i=1}^{n} l_i \leqslant \text{const}_n \text{Vol}\,V.$$

The above arguments allow one to assume the cycle V to be regular, and then the balls $B_v(R)$ in V for $R \leq \frac{1}{2}\text{sys}(V, f)$ have

$$\text{Vol } B_v(R) \geq A_{n-1} R^n.$$

This solves the problem for $n = 2$.

Furthermore let $n = 3$, let γ_1 be the shortest nontrivial (in $H_1(T^3, \mathbf{R})$) curve in V, and let γ_2 be the next shortest curve whose image $f(\gamma_2)$ is independent of $f(\gamma_1)$ in $H_1(T^3, \mathbf{R})$. As V is regular,

$$l_1^2 l_2 \leq \text{const Vol } V.$$

A stronger result would be

$$l_1 l_2^2 \leq \text{const Vol } V.$$

In fact, one can show this to be true, provided the *regular* cycle V is *homotopy equivalent* to T^3. This is seen by analyzing the balls $B_v(R) \subset V$ and their boundaries, for some point $v \in \gamma_2$ and $R \approx l_2/5$.

6.6. Systems of short based loops in aspherical manifolds. Let V be the same closed aspherical manifold as in §6.5, and let us try to find a system of "short" loops $\gamma_1, \cdots, \gamma_q$ with *a common* base point $v_0 \in V$ such that the subgroup generated by these loops in $\pi_1(V, v_0)$ is as large as possible. We shall use a regularisation V_δ^* of V, which is somewhat different from V^* of §6.5; namely, we start with some δ-extension $K_\delta(V) = K_\delta(V, V_0)$, for $\delta = \frac{1}{3}\text{sys}_1(V)$ and a finite ε'-net V_0 in V with a very small $\varepsilon' > 0$. We assume without loss of generality the map $J\colon V \to K_\delta(V)$ to be isometric (as the numbers ε and ε' of Lemma 6.3.A may be chosen as small as we wish), and we take a *connected component* V_δ^* of the (almost) minimal cycle in $K_\delta(V)$, which is homologous to $V \subset K_\delta(V)$. This V^* is a "geometric cycle" (see Remark 6.4.B′) which represents some *nonzero integral multiple* of the fundamental class of V. Furthermore, the balls in this V_δ^* have

$$\text{Vol } B_v(R) \geq A'_{n-1} R^n, \tag{6.5}$$

(see the proof of Theorem 6.4.A). Moreover, if some points $v_0 \in V \subset K_\delta(V)$ and $v_0^* \in V_\delta^* \subset K_\delta(V)$ are contained in the same δ-cube of the complex $K_\delta(V)$, then the corresponding minimal loops γ in V at v_0 and γ^* in V_δ^* at v_0 satisfy

$$\text{length}\,\gamma \leq \text{length}\,\gamma^* + \delta,$$

(see Corollary 6.3.B).

Thus the problem of finding short loops in V is reduced to the corresponding problem in the cycle V_δ^*, where we are aided by inequality (6.5).

As the cycle V_δ^* represents a *nonzero* multiple of the class $[V]$, the image of the fundamental group $\pi_1(V_\delta)$ under the map $V_\delta^* \to V$ has finite index in $\pi_1(V)$ and so we obtain the following.

6.6.A. Theorem. *There exist some loops $\gamma_1, \cdots, \gamma_q$ in V with a common base point $v_0 \in V$, which generate a subgroup of finite index in the group $\pi_1(V)$, such that the lengths l_i of γ_i satisfy*

$$l_1 = \operatorname{sys}_1(V, v_0) \leqslant 2(\operatorname{Vol} V/A'_{n-1})^{1/n}, \tag{6.6}$$

(*compare* (1) *of Theorem* 6.5.A), *and*

$$l_i \leqslant 3l_1 + 2^{n+1}\operatorname{Vol} V/l_1^{n-1}A'_{n-1}, \quad i = 2, \cdots, q, \tag{6.7}$$

(*Compare* (2) *of Theorem* 6.5.A).

Our next goal is to find as many as possible "independent" loops γ_i of lengths $l_i \leqslant \operatorname{const}_n(\operatorname{Vol} V)^{1/n}$. Let us agree to say that some loops $\gamma_1, \cdots, \gamma_q$ in V with a common base point $v_0 \in V$ are *dependent* if the subgroup which they generate in $\pi_1(V, v_0)$ is *almost nilpotent*, that is, it contains a nilpotent subgroup of finite index. One might use a different definition with another class of "small" groups, such as almost abelian or almost solvable groups. Our choice is motivated by the following version of Margulis' lemma on manifolds $\tilde V$ whose Ricci curvature is bounded below (compare [18]).

Let $\tilde V$ be a complete Riemannian manifold, and let $\gamma_1, \cdots, \gamma_q$ be some isometries of $\tilde V$, which generate a discrete subgroup Γ in the isometry group of $\tilde V$. Take a point $\tilde v_0 \in \tilde V$, and let

$$\delta = \min_{1 \leqslant i \leqslant q} \operatorname{dist}(\tilde v_0, \gamma_i(\tilde v_0)) > 0,$$

$$\delta_+ = \max_{1 \leqslant i \leqslant q} \operatorname{dist}(\tilde v_0, \gamma_i(\tilde v_0)) \geqslant \delta.$$

6.6.B. Theorem (*see* [32]). *There exists a positive constant $\varepsilon = \varepsilon(\dim \tilde V, C) > 0$ for $C = \delta_+/\delta$ such that the inequality*

$$\delta_+^2 \inf \operatorname{Ricci} V \geqslant -\varepsilon$$

implies that the group Γ is almost nilpotent.

Now let Π be an arbitrary group, and let $\Delta(\Pi^*)$ be the (infinite) simplex spanned by the elements $\pi \in \Pi^* = \Pi \backslash \mathrm{id}$, i.e., the simplicial complex whose k-simplices are $(k+1)$-tuples $(\pi_0, \cdots, \pi_k)$ for $\pi_i \in \Pi^*$, $i = 0, \cdots, k$. Let the group Π act on this complex by conjugation $\pi(\pi_0, \cdots, \pi_k) = (\pi\pi_0\pi^{-1}, \cdots, \pi\pi_k\pi^{-1})$ for all $\pi \in \Pi$, and say that some elements $\pi_0, \cdots, \pi_k$ in Π are *dependent* if they generate an almost nilpotent subgroup in Π. Denote the Π-invariant subcomplex in $\Delta(\Pi^*)$ by $Q = Q(\Pi^*) \subset \Delta(\Pi^*)$, whose k-simplices are spanned by $(k+1)$-tuples of dependent elements in Π.

Let the group Π act freely and isometrically on a Riemannian manifold (or pseudo-manifold) $\tilde{V}$. Take a point $\tilde{v} \in \tilde{V}$ and let $\pi_0, \cdots, \pi_k$ be all *systolic* isometries:

$$\operatorname{dist}(\tilde{v}, \pi_i(\tilde{v})) = \inf_{\pi \in \Pi^*} \operatorname{dist}(\tilde{v}, \pi(\tilde{v})), \quad i = 0, \cdots, k.$$

Thus we assign to each point $\tilde{v} \in V$ Π-equivariantly a k-simplex $\Delta_{\tilde{v}} = (\pi_0, \cdots, \pi_k)$ in $\Delta(\Pi^*)$, called the *systolic simplex* at $\tilde{v}$. Then by an obvious partition of unity argument we obtain a continuous Π-equivariant map, called α: $\tilde{V} \to \Delta(\Pi^*)$, which sends $\tilde{V}$ into the union of the systolic simplices $\Delta_{\tilde{v}}$ over all $\tilde{v} \in V$. We obtain, in particular, the following.

6.6.B′. Lemma. *Let for every point $\tilde{v} \in \tilde{V}$ the systolic (at $\tilde{v}$) isometries $\pi_0, \cdots, \pi_k$ are dependent. Then for every $\varepsilon > 0$ there exists a continuous Π-equivariant map α: $\tilde{V} \to Q$ sending each point $\tilde{v} \in V$ into some systolic simplex $\Delta_{\tilde{v}'} \subset Q$ for* $\operatorname{dist}(\tilde{v}', \tilde{v}) \leqslant \varepsilon$.

Examples. Suppose that every almost nilpotent subgroup of Π is contained in a *unique* maximal almost nilpotent subgroup. The following are such groups:

1. Almost nilpotent groups.

2. Subgroups of the fundamental groups of compact manifolds of negative curvature.

2′. Subgroups of the fundamental groups of complete manifolds V of negative curvature, which have

$$-\kappa_1^2 \leqslant \operatorname{Curvature}(V) \leqslant -\kappa_2^2, \quad \text{for } \kappa_1, \kappa_2 > 0,$$

and $\operatorname{Vol} V < \infty$.

3. Free products of groups in the above Examples 1, 2 and 2′.

The complex Q for the above group Π consists of the disjoint union of (finite or infinite) simplices Δ_N, each spanned by a maximal almost nilpotent subgroup $N \subset \Pi$ for all such $N \subset \Pi$. If the manifold $\tilde{V}$ in Lemma 6.6.B′ is connected, then the image of α is contained in one such symplex Δ_N, and so *all* systolic isometries *for all* points $\tilde{v}$ in V are contained in N. These isometries generate a *normal* subgroup in Π, and so there is *a nontrivial normal almost nilpotent subgroup in* Π.

The groups Π in the above Examples 2, 2′ and 3 do not contain such normal subgroups, unless they themselves are almost nilpotent. Therefore an isometric action of such a group Π on a connected manifold $\tilde{V}$ always possesses a system of *independent* systolic elements at some point $\tilde{v} \in \tilde{V}$.

Let us apply these considerations to our original problem of locating short independent loops in essential manifolds V. To be specific, we assume the group Π to be isomorphic to the fundamental group of a closed manifold V_0 of negative curvature, i.e., $\Pi = \pi_1(V_0)$, and then we consider an n-dimensional

Riemannian manifold V, for $2 \leqslant n \leqslant \dim V_0$, which admits a map f: $V \to V_0$ such that the image $f_*[V] \in H_n(V_0)$ does not vanish.

6.6.C. Theorem. *There exist two geodesic loops γ_0 and γ_1 at some point $v \in V$, whose images $f(\gamma_0)$ and $f(\gamma_1)$ are independent in the group $\Pi = \pi_1(V_0, f(v))$, such that*

$$\text{(6.8)} \qquad \begin{aligned} &\operatorname{length} \gamma_0 = \operatorname{sys}(V, v; f) \leqslant A''_{n-1} (\operatorname{Vol} V)^{1/n}, \\ &\operatorname{length} \gamma_1 \leqslant 2 \operatorname{length} \gamma_0, \end{aligned}$$

for some universal constant $A''_n > 0$.

Proof. We use a regularisation V^*_δ of V (compare the proof of Theorem 6.6.A), for which we get two independent systolic isometries $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of the covering $\tilde{V}^*_\delta$ at some point $\tilde{v} \in \tilde{V}^*$. The inequality

$$\operatorname{Vol} B_{\tilde{v}}(R) \geqslant A'_{n-1} R^n,$$

for the balls in $\tilde{V}^*_\delta$ shows that

$$\operatorname{dist}(\tilde{\gamma}_i(\tilde{v}), \tilde{v}) \leqslant B'_n (\operatorname{Vol} V_\delta)^{1/n},$$

for $i = 1, 2$ and some constant B'_n, and so the geodesic loops in V corresponding to these isometries satisfy the estimate (6.8).

6.6.C′. Corollary. *If*

$$\inf \operatorname{Ricci} V \geqslant -\kappa^2, \quad \textit{for } \kappa \geqslant 0,$$

then

$$\operatorname{Vol} V \geqslant \varepsilon'_n \kappa^{-n}, \quad \textit{for some } \varepsilon'_n > 0.$$

Indeed, if the loops γ_1 and γ_2 are sufficiently short, then Theorem 6.6.B applies.

Remark. This corollary can also be proved by the technique of simplicial norms (see [32]).

Let us give a general criterion for the existence of independent isosystolic isometries. Denote by $Q/\!/\Pi$ the *homotopy quotient* of the action of Π on the complex $Q = Q(\Pi)$. By taking an aspherical space $\tilde{K}$ with a free action of the groups Π, and dividing the product $Q \times \tilde{K}$ by the diagonal action of Π, we have

$$Q/\!/\Pi \overset{\text{def}}{=} Q \times \tilde{K}/\Pi.$$

Equivariant maps $\tilde{V} \to Q$ give rise to sections of the projection p: $Q/\!/\Pi \to \tilde{K}/\Pi = K(\Pi; 1)$, and so we come to the following.

6.6.D. Proposition. *Let a map f: $V \to K = K(\Pi; 1)$ give a nonzero class $h = f_*[V] \in H_n(K)$. If the homomorphism p_*: $H_n(Q/\!/\Pi) \to H_n(K)$ vanishes, then there exists a system of independent systolic loops $\gamma_0, \cdots, \gamma_k$ at some point $v \in V$.*

Recall that by definition

$$\text{length}\,\gamma_i = \text{sys}(V, v;\, f), \quad i = 0, \cdots, k,$$

and that the subgroup Γ in $\Pi = \pi_1(K, f(v))$ generated by the loops $f(\gamma_i)$ contains no nilpotent subgroups $N \subset \Gamma$, which have finite index in Γ.

6.6.D′. Example. Let V_0 be a complete manifold of nonpositive sectional curvature. Suppose that every noncontractible closed curve in V_0 is freely homotopic to a closed geodesic in V_0. This is true, for instance, if V_0 is compact, or if V_0 covers a compact manifold isometrically. Let $d = d(V_0)$ denote the maximum of the dimensions of those geodesically convex subsets U in the universal covering $\tilde{V}_0$ of V_0, which split isometrically to products

$$U = U' \times \mathbf{R}.$$

Proposition. *If $d > d$, then the homomorphism*

$$p_*\colon H(Q/\!\!/\Pi) \to H_n(V_0) = H_n(K(\Pi; 1)),$$

for $\Pi = \pi_1(V_0)$, vanishes.

Proof. Every almost nilpotent subgroup Γ in Π is almost Abelian. (See [20].) Moreover, there is a unique maximal Abelian subgroup $A \subset \Gamma$ of finite index such that the union of all flat A-invariant l-dimensional subspaces in $\tilde{V}_0$ for $l = \text{rank}\, A$ form a convex subset $U_\Gamma \subset \tilde{V}$ which splits isometrically into the product $U_\Gamma = U'_\Gamma \times \mathbf{R}^l$ where the slices $u' \times \mathbf{R}^l$, $u' \in U'_\Gamma$, are the A-invariant flat subspaces. This convex set U_Γ is Γ-invariant. If $\Gamma_1 \subset \Gamma_2$, then $U_{\Gamma_1} \supset U_{\Gamma_2}$ (see [20]).

Let $\bar{V}_0 \subset \tilde{V}_0$ be the union of the sets U_Γ over all almost nilpotent subgroups $\Gamma \subset \Pi$. Clearly the set $\bar{V}_0$ is Π-invariant and $\dim \bar{V}_0 \leqslant d$. Let us construct a continuous Π-equivariant map $\tilde{q}\colon Q \times \tilde{V}_0 \to \bar{V}_0$. We first take the barycenter b of each simplex $(\gamma_0, \cdots, \gamma_k)$ in Q, and then send the pair $(b, v) \in Q \times \tilde{V}_0$ for every $v \in \tilde{V}_0$ to the point u in u_Γ nearest v, where Γ is the subgroup generated by the isometries $\gamma_0, \cdots, \gamma_k$. We extend this map to the barycentric symplices $(b_0, \cdots, b_k)$ in Q by induction, by taking the geodesic cone from b_k over the map of the base $(b_0, \cdots, b_{k-1})$.

Obviously the map $\tilde{q}$ admits a Π-equivariant (geodesic) homotopy to the projection $\tilde{p}\colon Q \times \tilde{V}_0 \to \tilde{V}_0$, and so the map

$$p\colon Q/\!\!/\Pi = \big(Q \times \tilde{V}_0\big)/\Pi \to V_0 = \tilde{V}_0/\Pi$$

is homotopic to the map

$$q\colon (Q \times V_0)/\Pi \to \bar{V}_0/\Pi,$$

which sends the n-dimensional homology of $(Q \times \tilde{V}_0)/\Pi$ for $n > d \geqslant \dim \bar{V}_0/\Pi$ to zero. q.e.d.

From this proposition we can deduce the following.

6.6.D″. Theorem (*Compare Theorem* 6.6.C *and Corollary* 6.6.C′). *Let a closed n-dimensional Riemannian manifold V admit a continuous map f into the above manifold V_0 such that $f_*[V] \neq 0$. If $n > d$, then there exist geodesic loops $\gamma_0, \cdots, \gamma_k$ at some point $v \in V$, whose images are independent in the group $\Pi = \pi_1(V_0, f(v))$, such that*

$$\operatorname{length}\gamma_0 = \operatorname{sys}(V, v, f) \leqslant A''_{n-1}(\operatorname{Vol} V)^{1/n},$$
$$\operatorname{length}\gamma_i \leqslant 2 \operatorname{length}\gamma_0, \quad \text{for } i = 1, \cdots, k.$$

Furthermore, if Inf Ricci $V \geqslant -\kappa^2$, *then* $\operatorname{Vol} V \geqslant \varepsilon'_n \kappa^{-n}$.

Remark. The volume estimate $\operatorname{Vol} V \geqslant \varepsilon'_n \kappa^{-n}$ for locally symmetric manifolds V of *nonpositive curvature* is due to Kazdan-Margulis (see [51]; also see [31] and [32] for related results).

6.6.E. Freely independent loops. The independence of some elements in a group Π often implies that some related elements in Π are *freely independent*. Recall that a subset $\Pi' \subset \Pi$ is said to be *freely independent* if every k elements in Π' generate a free subgroup of rank k in Π.

Examples. Let Π be the fundamental group of a compact manifold V_0 of negative curvature, and $\pi_0 \neq \mathrm{id}$ be an arbitrary element in Π. Then there is an integer m such that the set of the conjugate elements $\pi_i \pi_0^m \pi_i^{-1}$, $\pi_i \in \Pi$, is freely independent, provided no two elements π_i and π_j, $i \neq j = 0, \cdots$, are dependent. *In particular, the normal subgroup in* Π *generated by the element* π_0^m *is free.* (Compare [25], [75], [57], [55].)

To see this we consider an arbitrary map α to V_0 of a connected surface S with boundary ∂S such that ∂S is sent to the closed geodesic γ_0 which represents the conjugacy class of π_0, and such that every component of the boundary ∂S goes to either m times γ_0 or $-m$ times γ_0. If the map of the relative fundamental "groups",

$$\alpha_* : \pi_1(S, \partial S) \to \pi_1(V_0, \gamma_0),$$

is injective, then the area of the map α is bounded below:

$$\operatorname{Area} \alpha \geqslant \operatorname{const} qm,$$

where q denotes the number of the components of the boundary ∂S of S, and the constant const > 0 depends only on V_0 and γ_0. Indeed, we may assume without loss of generality that γ_0 is a simple curve in V_0, and then it admits a tubular ε-neighborhood U_ε for some small $\varepsilon > 0$. The pull-back $\alpha^{-1}(U_\varepsilon) \subset S$ contains q components adjacent to the boundary components of S, and each of them has area at least $\frac{1}{2}\varepsilon m$ length γ_0.

Now as V_0 has negative curvature, every map α can be homotoped to some minimal map, for an example, to a harmonic map

$$\beta\colon (S, \partial S) \to (V_0, \gamma_0).$$

Then the Gauss curvature of the induced metric in S is everywhere $\leqslant -\kappa^2$, for

$$-\kappa^2 = \sup \text{Curvature}(V) < 0,$$

and by the Gauss-Bonnet theorem

$$|\chi(S)| \geqslant 2\pi\kappa^2 \text{ Area } \beta.$$

So the above Euler characteristic satisfies the following inequality:

$$|\chi(S)| \geqslant \text{const}'\, qm,$$

Recall that any relation between some q elements in Π, which are conjugate to $\pi_0^{\pm m}$, can be represented by a map $(S, \partial S) \to (V_0, \gamma_0)$, where S is a surface of genus zero, for which $|\chi(S)| = q - 1$. Thus

$$|q - 1| \geqslant \text{const}'\, qm,$$

and so there is no relations for $m \geqslant 2(\text{const}')^{-1}$. q.e.d.

Let us give a sharper (but somewhat weaker) freedom property of the group $\Pi = \pi_1(V_0)$. We denote the *pinching constant* of the manifold V_0 by $\rho = \rho(V_0)$:

$$\rho = \frac{\inf \text{Curvature } V_0}{\sup \text{Curvature } V_0} > 0.$$

6.6.E′. Proposition. *There exists a constant $C = C(\rho, \dim V_0) > 0$ such that for every two independent elements π_0 and π_1 in Π the elements $\pi = \pi_0^m$ and $\pi' = \pi_1 \pi_0^m \pi_1^{-1}$ are freely independent for every $m \geqslant C$.*

Proof. Let γ and γ' be the geodesics in the universal covering $\tilde{V}_0$, which are invariant under the isometries π and π' respectively. Let δ be the shortest geodesic segment joining two points $x \in \gamma$ and $x' \in \gamma'$ and orthogonal to both γ and γ'.

Take the two points x_+ and x_- on γ, for which

$$\text{dist}(x, x_+) = \text{dist}(x, x_-) = l_m = \tfrac{m}{2}\text{dist}(x, \pi_0(x)),$$

and also take the two points x'_+ and x'_- on γ', for which

$$\text{dist}(x', x'_+) = \text{dist}(x', x'_-) = l_m.$$

We denote by X_+ and X_- (respectively, X'_+ and X'_-) the two *disjoint* halfspaces in V_0, which are bounded by the "hyperplanes" formed by the geodesics normal to γ (respectively γ'), and the points x_+ and x_- respectively. If there is no intersections between the four halfspaces, then obviously the isometries π and π' are freely independent (compare [25], [75]). On the other hand, if some halfspaces, say X_+ and X'_+, do intersect, then by the standard comparison theorems (see [20]) the geodesic segments $(x, \pi_0 x)$ and $(x', \pi_1 \pi_0 \pi_1^{-} x')$ must be "close" one to another. This "closeness" is estimated by sup Curvature V_0 and m.

Now by Margulis' Lemma (see [18]) and the lower bound on the curvature of V_0, the "closeness" of these segments implies that the isometries π_0 and $\pi_1 \pi_0 \pi_1^{-1}$ are dependent. Then the isometries π_0 and π_1 are also dependent, and so we get a lower bound for m. q.e.d.

We refer the reader to the work of Heintze [43], where one finds the details of this argument, which is used by Heintze for an analogous problem.

The above proposition allows one to sharpen Theorem 6.6.C by requiring the loops $f(\gamma_1)$ and $f(\gamma_2)$ (see Theorem 6.6.C) to be freely independent. Thus we have

6.6.E″. Theorem. *Let f be a continuous map of a closed Riemannian manifold V to a closed manifold V_0 of negative curvature such that the fundamental class of V goes to a nonzero class, $0 \neq f_*[V] \in H_n(V_0)$. Then there are two loops γ' and γ'_1 at some point $v \in V$, whose f-images are freely independent in the group $\Pi = \pi_1(V_0)$, such that*

$$\operatorname{length} \gamma'_0 \leqslant C' \operatorname{sys}(V, v, f) \leqslant C' A''_{n-1} (\operatorname{Vol} V)^{1/n},$$
$$\operatorname{length} \gamma'_1 \leqslant \operatorname{length} \gamma'_0 + 4 \operatorname{sys}(V, v, f),$$

where the constant $C' > 0$ depends only on $\dim V$, $\dim V_0$, *and the pinching constant $\rho = \rho(V_0)$.*

Proof. Take $\gamma'_0 = \gamma_0^m$ and $\gamma'_1 = \gamma_1 \gamma_0^m \gamma_1^{-1}$ for the loops γ_0 and γ_1 of Theorem 6.6.C.

6.7. Systoles of 2-dimensional polyhedra. The isoperimetric inequality 6.2.B. applies (trivially) to arbitrary 1-dimensional sub-polyhedra of cubical δ-complexes K:

Any 1-dimensional sub-polyhedron L of K, for which

$$\operatorname{length} L = l \leqslant \tfrac{1}{3}\delta,$$

"bounds" a cone in K of area $\leqslant Al^2$, for some universal constant $0 < A < 10$. Therefore the regularization technique of §§6.3–6.6 applies to the following *homotopy Plateau problem*:

Find a 2-dimensional subspace in K of least area, which is not contractible to the 1-skeleton of K. (See Appendix 2 for an n-dimensional generalization.)

One obtains as before a universal lower bound for the area of such minimal subspaces, and arrive at the following isosystolic inequalities.

6.7.A. Theorem. *Let V be a compact connected 2-dimensional polyhedron with a piecewise Riemannian metric. Then the following hold*:

(a) $$\operatorname{sys}_1(V) \leqslant B(\operatorname{area} V)^2,$$

for some constant in the interval $0 < B \leqslant 100$, *unless the fundamental group* $\pi_1(V)$ *is free.*

(b) *If the fundamental group* $\pi_1(V)$ *is neither free nor a (nontrivial) free product, then there are some loops* $\gamma_1, \cdots, \gamma_q$ *at some point* $v \in V$, *which generate the group* $\pi_1(V)$, *such that*

$$l_1 = \operatorname{length} \gamma_1 = \operatorname{sys}_1(V, v) \leqslant B(\operatorname{Area} V)^{1/2},$$
$$l_i \leqslant B(\operatorname{Area} V)/l_1, \text{ for } i = 2, \cdots, q.$$

6.7.A′. Corollary. *Let V_0 be a closed manifold of negative curvature, and let* $\Gamma \neq \mathbf{Z}$ *be a finitely presented group which is not a (nontrivial) free product. Then the fundamental group* $\Pi = \pi_1(V_0)$ *contains at most finitely many conjugacy classes of subgroups, which are isomorphic to* Γ.

Proof. Let V be a two-dimensional polyhedron such that $\pi_1(V) = \Gamma$. Every injective homomorphism $\Gamma \to \Pi$ is induced by a continuous map $f: V \to V_0$. According to Thurston (see [74]) one can straighten the map f on all 2-simplices of V and thus obtain a new map $g: V \to V_0$, which is homotopic to f and has

$$\operatorname{Area} g \leqslant \operatorname{const} = \operatorname{const}(V, V_0).$$

The conjugacy class of the subgroup $f_*(\Gamma) = g_*(\Gamma) \subset \Pi$ is uniquely determined by the restriction of g to a set of loops in $g_*(V)$, which generate $g_*(\Gamma)$. As the systole $\operatorname{sys}_1(V)$ of the polyhedron V with the induced metric is greater than or equal to $\operatorname{sys}_1(V_0) > 0$, the group $g_*(\Gamma) \subset \Pi$ is determined by some loops in V of lengths $\leqslant \operatorname{const}'(V, V_0)$. There are at most finitely many of homotopy classes of such systems of loops in V_0.

Remark. This argument for surface groups Γ is due to Thurston [74].

7. Besikovič' lemma

Take an arbitrary Riemannian metric on the n-dimensional cube $C \approx I^n$. *Besikovič' lemma* (see [73], [14], [22], [5], [23]) claims the following *lower bound* for the total volume of this metric by the product of the distances $\operatorname{dist}(F_i, \bar{F}_i)$ between the opposite $(n-1)$-faces $(F_i, \bar{F}_i)$, $i = 1, \cdots, n$, of the cube:

$$\operatorname{Vol} C \geqslant \prod_{i=1}^{n} \operatorname{dist}(F_i, \bar{F}_i). \tag{7.1}$$

The inequality (7.1) sharpens the following classical theorem of Lebesgue (see [46]):

Let f: $C \to K$ be a continuous map of the n-cube to an arbitrary $(n-1)$-dimensional space K. Then there exists a pair of (opposite) points $x \in F_i$ and $\bar{x} \in \bar{F}_i$ for some $i = 1, \cdots, n$ such that $f(x) = f(\bar{x})$.

Derrick's proof of (7.1) (see [22]) only depends on the compressing property of mass*(= Vol for Riemannian metrics); that proof generalizes as follows.

Consider an orientable n-dimensional manifold (or pseudomanifold) W with boundary $V = \partial W$. Let this boundary be covered by $2n$ closed subsets ("faces") F_i' and $\bar{F}_i$ in V for $i = 1, \cdots, n$. We call the manifold W a "*cube*" *of dimension n and degree d*, and the manifold V a ∂ "*cube*", if there exists a continuous map of degree d of V to the boundary of the standard cube h: $V \to \partial I^n \approx S^{n-1}$ such that the pullbacks of the faces of the cube I^n equal the "faces" of $W \supset V$, $h^{-1}(F_i) = F_i'$ and $h^{-1}(\bar{F}_i) = \bar{F}_i'$ for $i = 1, \cdots, n$.

Observe that any two maps of a "cube" to the cube, which both send the "faces" F_i' and $\bar{F}_i'$ to the respective faces F_i and $\bar{F}_i$, are homotopic relative to the boundaries. Therefore all such maps have degree = degree ("cube").

Examples. (a) The standard cube is a "cube" of degree 1.

(b) Let W be a $4d$-gon in the plane with the edges e_i, $i = 1, \cdots, 4d$, and put

$$F_1' = \bigcup_{i \equiv 0 \,(\mathrm{mod}\, 4)} e_i, \quad \bar{F}_1' = \bigcup_{i \equiv 2 \,(\mathrm{mod}\, 4)} e_i,$$
$$F_2' = \bigcup_{i \equiv 1 \,(\mathrm{mod}\, 4)} e_i, \quad \bar{F}_2' = \bigcup_{i \equiv 3 \,(\mathrm{mod}\, 4)} e_i.$$

Then W is a 2-dimensional "cube" of degree d.

(c) The product of two "cubes" $(W'; F_i', \bar{F}_i')$ of degree d' and $(W''; F_j'', \bar{F}_j'')$ of degree d'' is a "cube" of degree $d'd''$:

$$\left(W' \times W''; F_i' \times W'', \bar{F}_i' \times W'', F_j'' \times W', \bar{F}_j'' \times W'\right).$$

(d) If a "face" F_1' of a cube is a (pseudo)manifold, then it is an $(n-1)$-dimensional "cube" of degree = degree (W). The "faces" of F_1' are the intersections $F_1' \cap F_i'$ and $F_1' \cap \bar{F}_i'$ for $i = 2, \cdots, n$.

7.1. A lower bound for the volume of a "cube". *Let an n-dimensional "cube" W of degree d be imbedded into some Banach space L: $W \hookrightarrow L$. Then the n-dimensional mass* of this cube is bounded below by the (induced) distances between the opposite faces*:

$$\text{(7.2)} \qquad \operatorname{mass}^*(W) \geq |d| \prod_{i=1}^{n} \operatorname{dist}\left(F_i', \bar{F}_i'\right).$$

In particular, every $(n-1)$-dimensional ∂ "cube" V of degree d with an arbitrary metric has

$$\text{Fill Vol}(V) \geqslant \text{Fill mass}^*(V) > |d| \prod^{n} \text{dist}(F_i', \bar{F}_i').$$

Proof. Take the following solid (cube) in $\mathbf{R}^n$:

$$I_0^n = \{(x_1, \cdots, x_n) \mid 0 \leqslant x_i \leqslant \delta_i = \text{dist}(F_i', \bar{F}_i')\}.$$

We claim that there is a map h_0: $W \to I_0^n$ sending the "faces" F_i' and $\bar{F}_i'$ to the respective faces of I_0^n such that h_0 is distance-decreasing relative to the l^∞-norm in $\mathbf{R}^n$:

$$\|(x_1, \cdots, x_n)\|_{l^\infty} = \max_{1 \leqslant i \leqslant n} |x_i|.$$

One can construct such a map by induction on n: if every pair of "faces" F_i' and $\bar{F}_i'$, $i = 1, \cdots, n$, has been already sent to the pair of the corresponding faces of the cube I_0^n with dilation $\leqslant 1$, then by the "compressing property" of the l^∞-norm, such a map extends to all of W with dilation $\leqslant 1$ (see §§1.1, 4.1).

Here is an obvious direct construction: take the following n functions on W:

$$x_i(w) = \min(\delta_i, \text{dist}(v, F_i')), \quad i = 1, \cdots, n.$$

The map $X(v) = (x_1(v), \cdots, x_n(v))$ is the required map h_0.

The map h_0 has degree d and is distance-decreasing. Therefore it is mass*-decreasing, and so

$$\text{mass}^* W \geqslant d \, \text{mass}^* I_0^n = d \prod_{i=1}^{n} \delta_i.$$

7.1.A. Corollary. *The $(n-1)$-dimensional mass* of $V = \partial W \subset L$ satisfies*

$$\text{mass}^* V \geqslant 2d \sum_{i=1}^{n} \left(\delta_i^{-1} \prod_{j=1}^{n} \delta_j \right).$$

There is another proof of Besikovič' lemma, which is due to Almgren [5] and depends on the coarea formula; namely, the pullbacks of the distance function

$$\delta(w) = \text{dist}(w, F_1')$$

are also "cubes":

$$W(t) = \delta^{-1}(t), \quad t \in [0, \delta_1 = \text{dist}(F_1', \bar{F}_1')],$$

which have dimension $n-1$ and degree d. The coarea formula implies

$$\text{mass}^* W \geqslant \int_0^{\delta_1} \text{mass}^* W(t) \, dt,$$

and the proof follows by induction.

Almgren's proof yields the following generalization of Besikovič' lemma. Take the product of a "cube" W of degree d by an arbitrary closed oriented manifold (or pseudomanifold) W_0 of dimension k. We embed the product $W \times W_0$ into some Banach space L, and denote by $\text{mass}^*(d[W_0])$ the lower bound of the k-dimensional masses of those k-dimensional cycles (or sub-pseudomanifolds) in $W \times W_0$, which go under the projection $W \times W_0 \to W_0$ to the d times the fundamental class of W_0.

7.1.B. *The $(n+k)$-dimensional mass* of the product $W \times W_0 \subset L$ satisfies*

$$(7.3)\quad \text{mass}^*(W \times W_0) \geqslant \text{mass}^*(d[W_0]) \prod_{i=1}^{n} \text{dist}(F_i' \times W_0, \bar{F}_i' \times W_0).$$

Proof. We construct as before a distance-decreasing map $h_0\colon W \times W_0 \to I_0^n$ sending the products $F_i' \times W_0$ and $\bar{F}_i' \times W_0$ in $W \times W_0$ to the corresponding faces of the solid (cube) I_0^n. The pullbacks $h^{-1}(x)$ of generic points $x \in I_0^n$ are cycles in $W \times W_0$, which project to $d[W_0] \in H_k(W_0)$. By the coarea inequality we obtain

$$\text{mass}^*(W \times W_0) \geqslant \int_{I_0^n} \text{mass}^*(h^{-1}(x))\, dx$$

$$\geqslant \inf_{x \in I^n} \text{mass}^*(h^{-1}(x)) \prod_{i=1}^{n} \delta_i \geqslant \text{mass}^*(d[W_0]) \prod_{i=1}^{n} \delta_i .$$

Example. Take a closed surface W_0 and consider an arbitrary Riemannian metric on the product $W_0 \times [0,1]$. Then there exists a smooth surface W_0' in $W_0 \times [0,1]$, which is homologous to $W_0 \times 0$ and has

$$\text{area}(W_0') \leqslant \text{Vol}(W_0 \times [0,1])/\text{dist}(W_0 \times 0, W_0 \times 1).$$

In fact, one has the *strict* inequality unless $W_0 \times [0,1]$ is an *isometric* product.

7.2. Conformal Besikovič' lemma. The Besikovič' lemma for Riemannian metrics on a "cube" was sharpened by Derrick [22] with the following classical notion of *p-distance* for $p \in [1, \infty)$.

For two subsets W_1 and W_2 in a Riemannian manifold W and a function $\varphi \geqslant 0$ on W, we introduce $\text{dist}_\varphi(W_1, W_2)$ as the lower bound of the integrals $\int_\gamma \varphi\, d\gamma$ over all curves γ in W joining some points w_1 and w_2 in W_1 and W_2 respectively. Then we consider those functions φ on W for which $\int_W \varphi^p\, dv \leqslant 1$ for a given number $p \geqslant 1$, and we define $\text{dist}^{(p)}(W_1, W_2)$ to be the upper bound of $\text{dist}_\varphi(W_1, W_2)$ over all functions $\varphi \geqslant 0$, for which $\int_W \varphi^p \leqslant 1$. If $\text{Vol}\, W = 1$, then

$$\text{dist}^{(p)} \geqslant \text{dist}^{(q)}, \quad \text{for } q \leqslant p,$$

and $\text{dist}^{(p)} \to \text{dist}$ as $p \to \infty$.

The most interesting p-distance is $\mathrm{dist}^{(n)}$ for $n = \dim W$, as it is a conformal invariant of W. If one of the sets W_i, $i = 1, 2$, is discrete (or, more generally, has conformal capacity zero; see [48]), then $\mathrm{dist}^{(n)}(W_1, W_2) = \infty$. However, $\mathrm{dist}^{(n)}(W_1, W_2) < \infty$ if both sets W_1 and W_2 have positive topological dimensions. The proof of these facts is straightforward.

7.2.A. (*Bes Conf*). *The distances* $\mathrm{dist}_i^{(n)}$ *between the opposite "faces' of an arbitrary n-dimensional "cube" W of degree d satisfy*

$$\prod_{i=1}^{n} \mathrm{dist}_i^{(n)} \leqslant |d|^{-1}.$$

Proof. Let $\varphi_i \geqslant 0$, $i = 1, \cdots, n$, be some functions on W, for which $\int_W \varphi_i^n \leqslant 1$, and whose distances $\mathrm{dist}_{\varphi_i}$ between the respective faces F_i' and $\bar{F}_i'$ are $\delta_1, \cdots, \delta_i, \cdots, \delta_n$. For each $i = 1, \cdots, n$ there exists a $(\mathrm{dist}_{\varphi_i})$-decreasing map h_i: $W \to [0, \delta_i]$ such that $h_i \mid F_i' \equiv 0$ and $h_i \mid \bar{F}_i' \equiv \delta_i$. The maps h_i, $i = 1, \cdots, n$, send W onto the solid $I_0^n = \times_i [0, \delta_i]$ by the map $\bar{h} = (h_1, \cdots, h_n)$ of degree d, whose Jacobian at every point $w \in W$ satisfies

$$|\mathrm{Jac}(w)| \leqslant \prod_{i=1}^{n} \varphi_i(w) \leqslant \frac{1}{n} \sum_{i=1}^{n} \varphi_i^n(w).$$

Therefore

$$d \prod_i \delta_i = d \operatorname{Vol} I_0^n \leqslant \int |\mathrm{Jac}(w)| dw \leqslant \frac{1}{n} \sum_{i=1}^{n} \int \varphi_i^n(w) \leqslant 1.$$

Remark. Inequalities (7.2) and (7.3) also hold if dist_i is replaced by dist_i^n, provided mass W is normalized to be one.

7.3. Besikovič' lemma for "simplices". Take a closed manifold V covered by some subsets ("faces") $F_1, \cdots, F_q$ in V. Suppose that the distances $\delta_j(v) = \mathrm{dist}(v, F_j)$ for every point $v \in V$ are restricted by some inequalities, that is, we are given a subset $\tilde{\Delta}$ in the Euclidean space $\mathbf{R}^q$ and require the vector $(\delta_1(v), \cdots, \delta_q(v) \in \mathbf{R}^q$ to belong to $\tilde{\Delta}$ for all $v \in V$. For example, we have used the covering of ∂ "cube" manifolds by $2n$ faces F_i' and $\bar{F}_i'$, $i = 1, \cdots, n$, such that

$$\mathrm{dist}(v, F_i') + \mathrm{dist}(v, \bar{F}_i') \geqslant \delta_i = \mathrm{dist}(F_i', \bar{F}_i').$$

Suppose that V is filled in by a manifold W with boundary $\partial W = V$. Then the map $\tilde{h}$: $W \to \mathbf{R}^q$ with the coordinates $\mathrm{dist}(w, F_j)$: $W \to \mathbf{R}$, $j = 1, \cdots, q$, expands the volume of W by a factor $\leqslant (q/n)^{n/2}$ for $n = \dim W$. Indeed, the differential of our map, say $A = A_w$: $\mathbf{R}^n \to \mathbf{R}^q$, has at every point $w \in W$

$$\text{Jacobian} = [\mathrm{Det}(A^*A)]^{1/2} \leqslant [n^{-n}(\operatorname{Trace} A^*A)^n]^{1/2},$$

where Trace A^*A = Trace AA^* = (sum of the squares of the l^2-norms of the rows of the matrix A) $= q$. Thus we may estimate the filling volume of V by the filling volumes of some cycles in the region $\tilde{\Delta} \subset \mathbf{R}^q$.

There is at least one example (besides the case of the cube) where one gets in this way a *sharp* estimate for Fill Vol(V); namely, take the n-dimensional simplex Δ and let $h\colon V \to \partial\Delta$ be a map of degree d. We take the pullbacks of the $(n-1)$-faces of Δ for $F_j, j = 1,\cdots,n+1 = q$, and claim the following.

7.3.A. Simplex inequality. *If*

$$\sum_{j=1}^{n+1} \delta_j(v) = \sum_{j=1}^{n+1} \operatorname{dist}(v, F_j) \geqslant \delta,$$

for all $v \in V$, *then*

$$\text{Fill Vol } V \geqslant \mu_n d\delta^n,$$

where μ_n *is the volume of the regular Euclidean n-simplex of the unit hight.*

Proof. The region $\tilde{\Delta} \subset \mathbf{R}^{q=n+1}$ consists of the vectors in $\mathbf{R}^q$ with positive components $x_j \geqslant 0$, for which $\Sigma_{j=1}^q x_j \geqslant \delta$, and so the (multiple) image of the map $\tilde{h} = W \to \mathbf{R}^q$ has a volume greater than d times the volume of the Euclidean simplex $\{x_j \geqslant 0, \Sigma_{j=1}^q x_j = \delta\}$.

Example. Take a $3d$-gon V, that is, a circle divided into $3d$ edges $e_1,\cdots,e_{3d}$. Suppose that for some given metric in V, every point $v \in V$ satisfies

$$\operatorname{dist}(v, e_i) + \operatorname{dist}(v, e_j) + \operatorname{dist}(v, e_k) \geqslant \delta,$$

for all triples of edges (e_i, e_j, e_k) which have $|i-j| \equiv |i-k| \equiv |j-k| \equiv 1 \pmod 3$. Then

$$\text{Fill Vol}(V) \geqslant d\delta^2/\sqrt{3}.$$

Remark. Observe that the filling radius of any ∂ "simplex" manifold V, for which $\Sigma_1^{n+1}\delta_j \geqslant \delta$ and $d > 0$, satisfies

$$\text{Fill Rad } V \geqslant \delta/2n.$$

Indeed every "simplex" W with the boundary $\partial W = V$ contains a point $w \in W$, for which

$$\operatorname{dist}(w, F_1) = \operatorname{dist}(w, F_2) = \cdots = \operatorname{dist}(w, F_{n+1}) = \varepsilon,$$

as an elementary topological argument shows. Then $2n\varepsilon \geqslant \delta$ and so

$$\varepsilon = \operatorname{dist}(w, \partial W) \geqslant \delta/2n.$$

7.4. Besikovič lemmas for closed manifolds. Consider a Riemannian manifold V, and for a homology class $\alpha \in H_k(V; \mathbf{Z})$, denote by Vol α the lower bound of the volumes of the integral singular cycles which represent α. In the same way we introduce the volume $\text{Vol}_{\mathbf{R}}$ on the *real* homology $H_k(V; \mathbf{R})$.

Observe that $\mathrm{Vol}_{\mathbf{R}}\,\alpha \leqslant \mathrm{Vol}\,\alpha$ for all $\alpha \in H_k(V; \mathbf{Z})$, but the equality may not, in general, hold. However, $\mathrm{Vol} = \mathrm{Vol}_{\mathbf{R}}$ does hold for the homology classes of codimension one in closed orientable manifolds (see [61] and [37] for additional information and references).

If the manifold V is compact (with or without boundary), then the function $\mathrm{Vol}_{\mathbf{R}}$ is a norm on the finite dimensional vector space $H_k(V; \mathbf{R})$. This norm induces a natural (flat Finsler) metric on the *Jacobi variety*, $J_k(V) = H_k(V; \mathbf{R})/H_k(V; \mathbf{Z})$, and we are interested in the total r-dimensional measure (volume) of the variety $J_k(V)$ for $r = \dim J_k(V)$. This is the measure of a fundamental domain of the lattice $H_k(V; \mathbf{Z}) \subset H_k(V; \mathbf{R})$, and so it depends on a particular choice (normalization) of the Haar measure in the Banach space $[H_k(V; \mathbf{R}), \|\ \| = \mathrm{Vol}_{\mathbf{R}}]$. To be specific we shall remain with the mass* in this space.

The total measure mass* $J_k(V)$ controls the asymptotic behavior of the number $N_k(R)$ of those integral k-dimensional homology classes in V which can be represented by cycles of volume $\leqslant R$. Namely

$$N_k(R)/R^r \to \mu_k^*/\text{mass}^*\, J_k(V), \quad \text{as } R \to \infty,$$

where $r = \operatorname{rank} H_k(V)$, and where μ_k^* denotes the mass* of the unit ball in the Banach space $H_k(V; \mathbf{R})$. Recall that $2^r/r! \leqslant \mu_k^* \leqslant 2^r$.

Observe that any *upper* bound on mass* $J_k(V)$ now gives a *lower* bound on the asymptotic number of "small" (in particular, minimal) cycles in V. Unfortunately, we do not obtain in this way any lower bound on the volume of any *individual* k-dimensional cycle in V, unless $k = \dim V - 1$. However, such individual estimates are available for the *relative* 1-dimensional homology of a "cube", $H_1(W, F_i' \cup \bar{F}_i')$, by Besikovič's lemma, and also for the first homology of some essential manifolds (see §6).

We shall establish below the following upper bound for the volumes of the complementary Jacobi varieties $J_k = J_k(V)$ and $J_{n-k} = J_{n-k}(V)$ for a closed orientable manifold V of dimension n:

$$(\text{mass}^*\, J_k) \times (\text{mass}^*\, J_{n-k}) \leqslant \text{const}^*(\mathrm{Vol}\, V)^r,$$

for $r = \dim J_k = \dim J_{n-k}$ and some universal constant $\text{const}^* = \text{const}^*(n, r)$. This inequality is implicit in the work of Blatter [15] and also in [37].

We shall use the method of Blatter which leads to a stronger version of the above inequality; namely, for the given metric g_0 on V we consider all conformally equivalent metrics $g = \varphi^2 g_0$ for which $\mathrm{Vol}(V, g) = \int_V \varphi^n\, dv \leqslant 1$. Every such metric g gives rise to a norm $\mathrm{Vol}_{\mathbf{R}}^g$ on homology $H_k(V; \mathbf{R})$, and we take the *upper* bound of these norms over all conformal metrics g of volume $\leqslant 1$. This upper bound is finite (the proof is straightforward), and so it is a conformally invariant norm, called the *conformal volume*, on the homology $H_k(V; \mathbf{R})$. We denote by conf* J_k the mass* of the Jacobi variety J_k with this conformal norm, and claim the following.

7.4.A. Conformal inequality.

$$(\text{conf}^* J_k) \times (\text{conf}^* J_{n-k}) \leqslant \text{const}^*(n, r).$$

Proof. Recall that the *comass norm* of an exterior k-form ω on $\mathbf{R}^n$ is

$$\|\omega\| = \sup \omega(e_1, \cdots, e_k),$$

where the supremum is taken over all orthonormal frames of vectors $e_1, \cdots, e_k$ in $\mathbf{R}^n$. Then for *differential* forms ω on V, one has this norm $\|\omega\|(v)$ on every tangent space $T_v(V)$, $v \in V$, and defines the L_n*-comass norm* $\|\omega\|$, to be $\int_V((\|\omega\|(v))^p\, dv\,)^{1/p}$ for $p \in [1, \infty]$. Next, one restricts such an L_p-norm to closed k-forms, and takes the quotient norm on the cohomology $H^k(V; \mathbf{R}) =$ (Closed forms)/(Exact forms). For $p = \infty$ this norm on $H^k(V; \mathbf{R})$ is called the (L_∞)*-comass norm*, and according to Federer it is dual to the volume norm on the homology $H_k(V; \mathbf{R})$ (see [27], [37]). It follows that the L_p*-comass norm* on H^k for $p = n/k$ *is dual to the conformal volume norm on* H_k. In fact, for every form ω on V one has the conformal (possibly degenerate) metric $g = [\|\omega\|_{L_p}^{-p/n} \|\omega\|^{1/k}(v)]^2 g_0$, which has $\text{Vol}(V, g) = 1$, and one observes that the pointwise norm $\|\omega\|(v)$ relative to g equals $\|\omega\|_{L_p}$ for all $v \in V$. Thus the integral of ω over any k-dimensional chain c in V satisfies

$$\omega(c) = \int_c \omega \leqslant \|\omega\|_{L_p} \text{Vol}_g c.$$

This implies the "half" of the duality claim, namely, the inequality

$$h'(h) \leqslant \|h'\|_{L_p} \text{conf Vol}(h),$$

for all $h' \in H^k(V; \mathbf{R})$ and $h \in H_k(V; \mathbf{R})$. To prove the second (nontrivial) half of the duality statement,

$$\|h'\|_{L_p} \leqslant \sup_{0 \neq h \in H_k} \left[|h'(h)| / \text{conf Vol}(h) \right],$$

one takes a homology class $h \in H_k$ of conf Vol $h = 1$ and then the conformal metric g for which $\text{Vol}_g h =$ conf Vol $h = 1$. By Federer's volume-comass duality for the manifold (V, g), there exists a closed form ω on (V, g), which represents the class h' and has the L_∞-comass $\leqslant |h'(h)|$. This establishes the duality.

Warning. Our L_2-norm on k-forms for $n = \dim V = 2k$ is not, in general, equal to the L_2-norm of the Hodge theory, as the local comass norm $\|\omega\|(v)$ is not the l^2-norm on the space $\Lambda^k T_v(V)$, unless $k = 1$ or $k = n - 1$. However, the two L_2-norms are equivalent, and so the canonical isomorphism of the space of harmonies k-forms on V with the ordinary (Hodge) l^2-norm onto the cohomology $H^k(V;\mathbf{R})$ with the L_2-comass norm has distortion $\leqslant$ const $=$ const(n). This constant equals one for $n=2$ (as $k=1$).

Next we observe the comass inequality for exterior products of forms:

$$\|\omega_1 \wedge \omega_2\|_{L_1} \leqslant \mathrm{const}\|\omega_1\|_{L_p}\|\omega_2\|_{L_q},$$

for $1/p+1/q=1$ and some universal constant const $=$ const(deg ω_1, deg ω_2). This follows from the corresponding local inequality:

$$\|\omega_1 \wedge \omega_2\|(v) \leqslant \mathrm{const}\|\omega_1\|(v)\|\omega_2\|(v).$$

Observe that

$$\mathrm{const} < \frac{(\deg \omega_1 + \deg \omega_2)!}{(\deg \omega_1)!(\deg \omega_2)!},$$

and that const $=1$ for deg $\omega_1 = 1$ and deg $\omega_2 = n-1$.

Now the cup product of two cohomology classes $h_1' \in H^k$ and $h_2' \in H^{n-k}$ satisfies

$$|h_1' \cup h_2'| \leqslant \mathrm{const}\|h_1'\|_{L_p}\|h_2'\|_{L_q},$$

for $p=n/k$, $q=n/(n-k)$ and const $=$ const$(k, n-k)$. Therefore the Poincaré duality map $PD\colon H^k \to H_{n-k} = (H^{n-k})^*$ has norm $\leqslant$ const. We write H^k as the dual to the homology H_k with the conformal volume norm, and then we have the map

$$PD\colon (H_k)^* \to H_{n-k}$$

of norm $\leqslant$ const, which sends the dual lattice $[H_k(V;\mathbf{Z})]^* \subset (H_k)^*$ onto the lattice $H_{n-k}(V;\mathbf{Z}) \subset H_{n-k} = H_{n-k}(V;\mathbf{R})$. We take a basis $e_1,\cdots,e_2$ in $H_k(V;\mathbf{Z})$, and then we have the inequality for the dual basis $e_1^*,\cdots,e_r^*$ in $H_k(V;\mathbf{Z}))^*$:

$$\mathrm{mass}^*(e_1^* \wedge \cdots \wedge e_r^*) \geqslant (\mathrm{const})^{-r}\,\mathrm{mass}^*\, J_{n-k}(V),$$

since the mass* of the Jacobian equals the mass* of some integral basis. Finally, by the definition of mass* we have

$$\mathrm{mass}^*(e_1{}^* \wedge \cdots \wedge e_r^*) = [\mathrm{mass}(e_1 \wedge \cdots \wedge e_r)]^{-1},$$

and so

$$(\mathrm{mass}\, J_k(V))(\mathrm{mass}^*\, J_{n-k}(V)) \leqslant \mathrm{const}^r \leqslant \left[\frac{n!}{k!\,(n-k)!}\right]^r.$$

As mass $\leqslant r^{-r/2}$ mass*, we obtain the required conformal inequality with

$$\mathrm{const}^* \leqslant r^{r/2}2^{nr}.$$

Remarks and corollaries. (a) The constant const* admits a better estimate. For example, the inequality mass $\geqslant r^{n/2}$ mass* may be improved for our Banach spaces because of their "L_p-origin". For example, for $k = 1$ and $n = 2$ the conformal norm is Euclidean, and thus we come to Blatter's inequality for surfaces V of genus $g \geqslant 1$:

$$\left(\text{mass}^* J_1(V)\right)^2 = \left(\text{Vol } j_1(V)\right)^2 \leqslant 1.$$

By a theorem of Minkowski every (Euclidean) flat torus of volume $\leqslant 1$ and dimension $2g$ possesses a closed geodesic of length $< (2/\pi)((g+1)!)^{1/g} \approx 2g/(\pi e)$, and thus Blatter proves that every closed surface V of genus g possesses a nonzero *homology* class in $H_1(V, \mathbf{Z})$ of conformal (extremal) length $\lessapprox 2g/(\pi e)$. We have seen in §5.5 that Loewner's methods yields a *homotopy* class of conformal lengths $\lesssim \sqrt{\log g}$.

(b) If $n = 2k > 2$, then one can easily see that mass $\geqslant (\text{const}_n)^{-r}$ mass* and thus gets

$$(\text{conf}^* J_k)^2 \leqslant (\text{const}'_n)^r.$$

In order to get an upper bound of the mass of some individual Jacobian J_k for $k \neq n/2$ one needs additional topological *nondegeneracy* conditions imposed on V.

Example. Let $\mathbf{R}^r$ be the Euclidean space with a fixed basis, and let Φ be a symmetric m-linear form on $\mathbf{R}^r$, which is represented in the given basis by a homogeneous polynomial of degree m. One assigns to each *nonzero* monomial in Φ,

$$a_{i_1\cdots i_r} x_1^{i_1} \cdots x_1^{i_r}, \quad \text{for } \sum_{j=1}^{r} i_j = m \quad \text{and } a_{i_1\cdots i_r} \neq 0,$$

the vector with integer components $(i_1, \cdots, i_r) \in \mathbf{Z}^r \subset \mathbf{R}^r$, and one calls the *Newton polyhedron* of Φ (relative to the given basis) the convex hull of those vectors which correspond to all nonzero monomials in Φ. The form Φ is said to be *nondegenerate relative to a given basis in* $\mathbf{R}^r$ if the Newton polyhedron contains a small vector $\bar{\varepsilon}$ with r equal components: $\bar{\varepsilon} = (\varepsilon, \cdots, \varepsilon) \in \mathbf{R}^r$ for some $\varepsilon > 0$. Furthermore, the form Φ is said to be *nondegenerate on* $\mathbf{R}^r$ if it is nondegenerate relative to every basis in $\mathbf{R}^r$.

Observe that this definition agrees with the ordinary conception of a nondegenerate quadratic form for $m = 2$.

If k is an even number and $\dim V = n = mk$ for an integer m, then the cup product on the cohomology $H^k(V; \mathbf{R})$ defines a *symmetric* m-form on $H^k(V; \mathbf{R})$.

7.4.B. Proposition. *If the cup product form is nondegenerate, then the formal mass* of the Jacobian J_k satisfies*

$$\mathrm{conf}^* J_k \leqslant \mathrm{const} = \mathrm{const}(k, m, r), \quad \textit{for } r = \dim J_k = \operatorname{rank} H_k(V).$$

Proof. We shall establish a more general result; namely, the inequality

$$\prod_{i=1}^{s} \left(\mathrm{conf}^* J_{k_i}\right)^{m_i} \leqslant \mathrm{const} = \mathrm{const}(k_i, m_i, r_i), \tag{7.4}$$

for $\Sigma_{i=1}^{s} k_i m_i = n$ under the following nondegeneracy assumption on the cup product form on the cohomology groups $H^{k_1}, \cdots, H^{k_s}$, whose ranks (over $\mathbf{R}$) are denoted by $r_1, \cdots, r_s$ respectively.

Take some bases

$$\left\{e_1^1, \cdots, e_{r_1}^1\right\} \text{ in } H^{k_1}, \cdots, \left\{e_1^s, \cdots, e_{r_s}^s\right\} \quad \text{in } H^{k_s}.$$

The cup product of some m_i elements among the vectors $e_1^i, \cdots, e_{r_i}^i$ in H^{k_i} is uniquely determined (up to $\pm$ sign for k_i odd) by the multiplicities of the entries e_j^i in this product. In other words, every such product is determined by a unique integral vector $M \subset \mathbf{Z}^{r_i}$ whose component M_j, $j = 1, \cdots, r_i$, is the multiplicity of e_j^i in the given product. We denote this product by $E_i^M \in H^{m_i k_i}$, and then introduce the *Newton polyhedron of the cup product form on the spaces* H^{k_i}, $i = 1, \cdots, s$ (this form has degree m_i on H^{k_i}), as the convex hull of those vectors $\overline{M} \in \mathbf{R}^r$ with nonnegative integer components

$$\overline{M} = (M_1, \cdots, M_s) \in \mathbf{Z}^{r_1} \oplus \cdots \oplus \mathbf{Z}^{r_s} \subset \mathbf{R}^r, r = r_1 + \cdots + r_s,$$

(where each vector M_i has the sum of the components equal m_i), for which the total cup product

$$E_1^{M_1} \cup \cdots \cup E_s^{M_s} \in H^n(V; \mathbf{R}) \approx \mathbf{R}$$

does not vanish.

The cup product form is said to be *nondegenerate relative to the bases* $\{e_j^i\}$ in $H^{k_i}, j = 1, \cdots, r_i$, $i = 1, \cdots, s$, if the Newton polyhedron contains some positive multiple of the vector

$$\Big(\underbrace{m_1, \cdots, m_1}_{r_1}, \cdots, \underbrace{m_i, \cdots, m_i}_{r_i}, \cdots, \underbrace{m_s, \cdots, m_s}_{r_s}\Big) \in \mathbf{R}^r.$$

Finally, we say the cup product form is *nondegenerate* if the above condition is satisfied for all systems of bases in the groups H^{k_i}. *We claim that inequality* (7.4) *holds under this nondegeneracy condition.*

To show that we take bases $\{e_1^i,\cdots,e_{r_i}^i\}$ in the *integral* cohomology groups $H^{k_i}(V;\mathbf{Z}) \subset H^{k_i} = H^{k_i}(V;\mathbf{R})$ for $i=1,\cdots,s$ with the following *quasiorthogonality property*:

$$\operatorname{mass}\left(e_1^i \wedge \cdots \wedge e_{r_i}^i\right) \geqslant \operatorname{const}_i \|e_1^i\| \times \cdots \times \|e_{r_i}^i\|,$$

where each space H^{k_i} is equipped with the respective conformal volume norm. By the elementary geometry of numbers, such a basis exists in every lattice $H^{k_i}(V;\mathbf{Z}) \subset H^{k_i}$ with some universal constant $\operatorname{const}_i = \operatorname{const}(r_i)$.

Consider the linear form $\Sigma_{i=1}^{s} \Sigma_{j_i=1}^{r_i} m_i l_{j_i}$ in variables l_{j_i}, and let us estimate a lower bound of the value of this form at $\{l_{j_i}\} = \{\log\| e_{j_i}(i)\|\}$. By the nondegeneracy condition this form is a positive combination (with some universal coefficients) of the forms $\Sigma_{i=1}^{s} \Sigma_{j_i=1}^{r_i} M_{j_i} l_{j_i}$, where $M_i = (M_{j_1},\cdots,M_{jr_i}) \in \mathbf{Z}^{r_i}$ are the multiplicities (exponents) of the "monomials" $E_i^{M_i} \in H^{m_i k_i}$ for which the cup product $E_1^{M_1} \cup \cdots \cup E_s^{M_s}$ is nonzero. Since this product is an integer (multiple of the fundamental class of V), its absolute value is at least one. Therefore

$$\exp \sum\sum M_{j_i} \log\left\|e_{j_i}^i\right\| \geqslant \operatorname{const} = \operatorname{const}(n).$$

Using this bound we also get some bound on the (universal combination) form $\Sigma\Sigma\, m_i \log\| e_{j_i}^i \|$ and thus the inequality

$$\prod_{i=1}^{s} \left[\operatorname{mass}\left(e_1^i \wedge \cdots \wedge e_{r_i}^i\right)\right]^{m_i} \geqslant \operatorname{const}(k_i, m_i, r_i),$$

which is equivalent, by the mass-mass* duality, to the inequality (7.4).

This argument, together with the use of integral quasiorthogonal bases, also shows that the existence of some cohomology classes

$$h'_{\mu_i} \subset H^{k_i}, \quad \mu_i = 1,\cdots,m_i,\, i = 1,\cdots,s,$$

with a *nonzero cup product*,

$$0 \neq \bigcup_{i=1}^{s} \bigcup_{\mu_i=1}^{m_i} h'_{\mu_i} \in H^n(V),$$

yields the *existence of some nonzero integral homology classes*, $H_{\mu_i} \in H_{k_i}(V;\mathbf{Z})$, *for which*

$$\prod_{i=1}^{s} \prod_{\mu_i=1}^{m_i} \operatorname{Conf\,Vol}_{\mathbf{R}} h_{\mu_i} \leqslant \operatorname{const}(k_i, m_i, r_i).$$

As a corollary we obtain for these classes the following.

7.4.C. Stable isosystolic inequality (compare [37], [11]).

$$\prod_{i=1}^{s} \prod_{\mu_i=1}^{m_i} \mathrm{Vol}_{\mathbf{R}}\, h_{\mu_i} \leqslant \mathrm{const}(k_i, m_i, r_i)\mathrm{Vol}\, V.$$

This inequality is unsatisfactory for two reasons. First, it is "stable" as it concerns the *real* volume $\mathrm{Vol}_{\mathbf{R}}$ rather than the actual volume on the *integral* homology. Secondly, the constant depends on $r_i = \mathrm{rank}\, H_{k_i}(V)$.

7.4.C′. Generalizations. There is a possible direction for improving the stable inequality; namely, one may try some nondegeneracy conditions on higher cohomology (Massey's) products. These reduce to the following operations on differential forms on V (see [71]):

(1) exterior products $\omega_1 \wedge \omega_2$,

(2) inversion of the exterior differentiation, that is, solving the equation $dx = \omega$.

To deal with (2), one needs some apriori estimates on relevant norms of solutions x of the equation $dx = \omega$. In particular, to keep $\inf_x \|x\|_{L_\infty}/\|\omega\|_{L_\infty}$ over all k-forms x for which $dx = \omega$, by the comass-volume duality one needs a bound on the *isoperimetric constant* in dimension k, that is, the upper bound of Fill $\mathrm{Vol}(z)/\mathrm{Vol}(z)$ over all k-dimensional cycles in V, which are homologous to zero. We shall approach this problem (see Theorem 7.5.C) in the case of the one-dimensional homology.

Let us indicate a similar problem where the higher products interract with the isoperimetric constants (compare [37]). Take a Riemannian manifold V homeomorphic to S^3, and let us introduce the "area" of V as the upper bound of areas of surfaces V_0 homeomorphic to S^2, for which there exists a *noncontractible* map $f\colon V \to V_0$ decreasing the areas of all surfaces in V.

Denote by Is_1 the first isoperimetric constant of V. By definition of Is_1, all oriented closed curves S in V have Fill $\mathrm{Vol}(S \subset V) \leqslant \mathrm{Is}_1 \mathrm{length}\, S$. We claim the following "*area*" *inequality*:

$$(\text{“area” } V)^2 \leqslant \mathrm{Is}_1 \mathrm{Vol}\, V.$$

Proof. As the map f is area-decreasing, the coarea formula yields the following relation for the lengths of the pullbacks $f^{-1}(v_0)$, $v_0 \in V_0$,

$$\int_{V_0} \mathrm{length}[f^{-1}(v_0)]\, dv_0 \leqslant \mathrm{Vol}\, V.$$

Therefore there is a (generic) point in V_0, whose pullback S is a curve of length $\leqslant \mathrm{Vol}\, V/\mathrm{area}\, V_0$. Thus curve S bounds an oriented surface A of area $\leqslant$ $\mathrm{Is}_1 \mathrm{length}\, S$. Since the map f is non-contractible, it has a nonzero Hopf invariant and so the map $f\colon A \to V_0$ is surjective. Hence area $A \geqslant$ area V_0, and then

$$(\text{area } V_0)^2 \leqslant \text{Is}_1 \text{ Vol}.$$

Remark. One proves in the same way that if all curves S in V have filling *radii* $\leqslant R_0$, then any *distance-decreasing* map $V \to V_0$ is contractible, provided $R_0 \leqslant \frac{1}{2}\text{Diam } V_0$. Notice that the number R_0 can be estimated for manifolds of *positive scalar curvature* $\geqslant K > 0$; namely, all curves S in $V \approx S^3$ have

$$\text{Fill Rad}(S \subset V) \leqslant \pi\sqrt{2}/K$$

(see [39]).

Jacobians of homogeneous manifolds. If V admits an isometric action of a compact connected Lie group G, then every closed form ω on V averages to a G-invariant form $\bar{\omega} \sim \omega$, which has $\|\bar{\omega}\|_{L_p} \leqslant \|\omega\|_{L_p}$ for all $p \geqslant 1$. Thus the evaluation of the L_p-norms on cohomology of *homogeneous* manifolds (G is transitive) reduces to a purely algebraic (local) problem. Then by duality one reconstructs relevant norms on homology.

Example (*Lawson*). Let V be the product of m unit spheres S^k. If $k = 1$, then the volume norm on $H_{k=1}$ is Euclidean. However, for $k \geqslant 2$ this norm is the l^∞-norm relative to the natural basis in $H_k(V; \mathbf{Z})$. Therefore integral combinations of the basic spheres $S_i^k \subset V = S_1^k \times \cdots \times S_m^k$ are absolutely volume-minimizing in their respective homology classes. In fact, these are the *only* absolutely minimizing cycles for $k \geqslant 3$.

This example shows that no bound on mass$^*J_k$ leads directly to any interesting information on the number of minimal k-dimensional subvarieties in V for $k \geqslant 2$. However, such information can probably be obtained for some manifolds with large fundamental groups.

For example, let V_0 be a compact manifold of nonpositive curvature, and let $f\colon V \to V_0$ be a map of positive degree. Let $A \subset \pi_1(V_0)$ be a maximal Abelian subgroup, $\tilde{V}_0 \to V_0$ the covering with $\pi_1(\tilde{V}_0) = A$, and $\tilde{V} \to V$ the induced covering of V. The manifold $\tilde{V}$ carries a k-dimensional homology class h_A for $k = \text{rank } A$, which goes to a multiple of the generator of the group $H_k(\tilde{V}_0) = \mathbf{Z}$. We realize this class by a minimal subvariety in V, and denote by $M(A) \subset V$ the projection of this variety to V. Now if two such subvarieties $M(A_1)$ and $M(A_2)$ in V coincide, then the subgroups A_1 and A_2 are conjugate in $\pi_1(V_0)$. It follows that every maximal flat torus in V_0 gives rise to a minimal subvariety in V.

In particular, if V_0 is a nonflat locally symmetric space of rank k, then the number of the flat tori T^k in V_0 grows exponentially for $\text{Vol}(T^k) \to \infty$.

Therefore

$$\operatorname{ent}_k V \overset{\text{def}}{=} \liminf_{t\to\infty} t^{-1} \log N_V(t) > 0,$$

where $N_V(t)$ denotes the number of geometrically different minimal subvarieties $M = M(A)$ in V of volume $\leqslant t$ for all subgroups $A \approx \mathbf{Z}^k$ in $\pi_1(V_0)$.

Conjecture. *If the universal covering $\tilde{V}_0$ of V_0 has no isometric Euclidean factors, then* $\operatorname{ent}_k V \geqslant \operatorname{const}_n(\operatorname{Vol} V)^{-k/n}$ *for* $n = \dim V$.

This conjecture is obviously true (by the length-area method of §5.5), if the map $f\colon V \to V_0$ is a conformal homeomorphism.

7.5. Counting short geodesics. Let us analyse our bound for mass* J_k for $k = 1$. First we indicate several geometric interpretations of the mass* of the Jacobian $J_1 = H_1(V; \mathbf{R})/H_1(V; \mathbf{Z})$, where the vector space $H_1(V; \mathbf{R})$ is equipped with the $\operatorname{Vol}_{\mathbf{R}}$-norm. This norm can be evaluated on an element $h \in H_1(V; \mathbf{Z})$ as follows. Take the shortest closed geodesic $\gamma = \gamma(h)$ in V, which is homologous to h (if there are several such geodesics, we choose one of them). Then

$$\operatorname{Vol}_{\mathbf{R}} h = \lim_{q\to\infty} q^{-1} \operatorname{length} \gamma(qh),$$

where $q = 1, 2, \ldots$ (see [37]).

Next we consider the number $N(R)$ of those geodesics $\gamma = \gamma(h)$ for all $h \in H_1(V; \mathbf{Z})$, for which

$$\operatorname{length} \gamma \leqslant R.$$

Then we have as $R \to \infty$

$$R^{-r} N(R) \to \mu^*[\mathrm{Tor}]/\text{mass}^* J_1,$$

where

$$r = \operatorname{rank} H_1(V; \mathbf{R}), \quad \mu^* = \text{mass}^* B_{H_1},$$

for the unit ball

$$B_{H_1} \text{ in } H_1 = H_1(V; \mathbf{R}),$$

and the integer $[\mathrm{Tor}] \geqslant 1$ denotes the order of the torsion subgroup in $H_1(V; \mathbf{Z})$ (compare [37]).

Recall that

$$2^r/r! \leqslant \mu^* \leqslant 2^r, \quad \text{for } r \geqslant 1,$$
$$3 \leqslant \mu^* \leqslant 4, \quad \text{for } r = 2.$$

Observe furthermore that the closed geodesics γ which correspond to *indivisible* elements h in the group $H^1(V; \mathbf{Z})$ are *prime* (i.e., not multiples of shorter geodesics). The percentage of indivisible elements in $H^1(V; \mathbf{Z})$ for $r \geqslant 2$ equals

$(\zeta(r))^{-1}$ for $\zeta(r) = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \cdots$, and so the asymptotic number for $R \to \infty$ of the above *prime* geodesics satisfies $R^{-r}NPr(R) \to \mu^*[\mathrm{Tor}]/\zeta(r)\mathrm{mass}^* J_1$.

Another geometric invariant of V related to $\mathrm{mass}^* J_1$ is the asymptotic volume of balls of radius $R \to \infty$ in the maximal abelian covering $\tilde{V}$ of V (whose deck transformation group is $H_1(V; \mathbf{Z})$); namely,

$$R^{-r}\mathrm{Vol}\, B(R) \to \mu^*[\mathrm{Tor}]\mathrm{Vol}\, V/\mathrm{mass}^* J_1.$$

Now we assume the cup product form (of degree $n = \dim V$) on the cohomology $H^1(V; \mathbf{R})$ to be nondegenerate. This condition implies the existence of n elements $h'_1, \cdots, h'_n$ in $H^1(V; \mathbf{R})$, whose cup product is nonzero. This is equivalent to the existence of a map onto the n-torus $V \to T^n$ of nonzero degree. Observe furthermore that these "nondegenerate" manifolds V have

$$r = \mathrm{rank}\, H_1(V; \mathbf{R}) \geqslant n = \dim V.$$

If $r = n$, then the nondegeneracy condition is *equivalent* to the existence of a map $V \to T^n$ of nonzero degree.

In the nondegenerate case we have an upper bound for $\mathrm{mass}^* J_1$ and thus asymptotic lower bounds for the geometric quantities $N(R)$, $NPr(A)$ and $\mathrm{Vol}\, B(R)$ for $R \to \infty$. Our upper bound for $\mathrm{mass}^* J_1$ can be somewhat sharpened with the "degree" of V, which is defined as the greatest common divisor of the degrees of all possible maps $V \to T^n$.

7.5.A. Theorem. *If (the cup product form on the one-dimensional cohomology of) the manifold V is nondegenerate, then*

$$\mathrm{mass}^* J_1 \leqslant \mathrm{const}(\deg)^{-r/n}(\mathrm{Vol}\, V)^{r/n},$$

for $\deg = \deg V =$ "degree" *and some universal constant* $\mathrm{const} = \mathrm{const}(n, r)$.

Proof. The only novelty here is the factor $(\deg)^{-r/n}$. It appears since every cup product of arbitrary *integral* classes $h'_1, \cdots, h'_h$ in $H^1(V; \mathbf{Z})$ is divisible by "deg" in the group $H^n(V; \mathbf{Z}) \approx \mathbf{Z}$.

Observe that $\mathrm{const}(n, n) = 1$, since

$$\|h'_1 \wedge \cdots \wedge h'_n\| \leqslant \|h'_1\| \times \cdots \times \|h'_n\|$$

for *one-dimensional* cohomology classes.

Also notice that the above theorem extends to all *Finsler* manifolds V with $\mathrm{mass}^* V$ substituted for $\mathrm{Vol}\, V$. Thus we obtain, for example, the following lower bound for the mass* of the balls $B(R)$ in the maximal Abelian covering V of a Finsler manifold V, for which $r = \mathrm{rank}\, H_1(V; \mathbf{R}) = n = \dim V$, which admits a map $V \to T^n$ of nonzero degree, and for which every map $V \to T^n$ has degree divisible by a number $\deg = \deg V \geqslant 1$:

$$\lim_{R \to \infty} R^{-n}\mathrm{mass}^* B(R) \geqslant \mu^*[\mathrm{Tor}]\deg.$$

This estimate is sharp for flat Finsler tori, for which [Tor] = deg = 1. For example, an arbitrary Finsler metric on T^2 satisfies

$$\lim_{R\to\infty} R^{-2}\,\mathrm{mass}^*\,B(R) \geqslant 3,$$

with the equality for the flat Finsler torus, whose universal covering carries the norm with the regular hexagon for the unit ball. (Compare §5.2.)

Observe that the numbers [Tor] and [deg] may assume arbitrary large values for manifolds V of dimension $\geqslant 4$ with Abelian fundamental groups.

Example. We start with the torus T^n, and take away an open regular neighborhood of the 1-skeleton of some triangulation of T^n. The boundary ∂ of the resulting manifold V' with boundary $\partial V' = \partial$ admits, for $n \geqslant 4$, an orientation reversing diffeomorphism $\partial \to \partial$ which induces the identity homomorphism on the (free) fundamental group of ∂. If we glue two copies of V' by this diffeomorphism, we get a manifold V, for which $\pi_1(V) = \pi_1(T^n) = \mathbf{Z}^n$, which admits a map $V \to T^n$ of degree 2, and moreover every map $V \to T^n$ has degree divisible by 2. In the same way one glues d-copies of T^n for any $d = 2, \cdots$, and gets a "nondegenerate" manifold V with $\deg V = d$ and $\pi_1(V) = \mathbf{Z}^n$.

The case of $r = \operatorname{rank} H_1(V;\mathbf{R}) < n = \dim V$. Let $f_0\colon V \to J_1 = J_1(V)$ be a continuous (Abel's) map, which induces the identity isomorphism

$$H_1(V;\mathbf{R}) \xrightarrow{\sim} H_1(J_1;\mathbf{R}) = H_1(V;\mathbf{R}).$$

Then we pass to the coverings of V and J_1 with

$$\text{Galois groups} = H_i(V;\mathbf{Z})/\text{Torsion} = H_1(J_1;\mathbf{Z}) \approx \mathbf{Z}^r,$$

and consider the covering map

$$\tilde f_0\colon \tilde V \to \tilde J_1 = H_1(V;\mathbf{R}).$$

The map $\tilde f_0$ is *isometric at infinity* (see [37]), *i.e.*,

$$\frac{\operatorname{dist}(\tilde v_1, \tilde v_2)}{\operatorname{dist}(\tilde f_0(\tilde v_1), \tilde f_0(\tilde v_2))} \to 1, \quad \text{for } \operatorname{dist}(\tilde v_1, \tilde v_2) \to \infty,$$

where

$$\operatorname{dist}(\tilde f_0(\tilde v_1), \tilde f_0(\tilde v_2)) \overset{\text{def}}{=} \operatorname{Vol}_{\mathbf{R}}(\tilde f_0(\tilde v_1) - \tilde f_0(\tilde v_2)).$$

It follows that for every l^∞-norm $\|\ \|_{l^\infty}$ in the space $H_1(V;\mathbf{R})$ for which $\operatorname{Vol}_{\mathbf{R}} > \|\ \|_{l^\infty}$, there exists a distance-decreasing map $\tilde f\colon \tilde V \to H_1(V;\mathbf{R})$ within finite distance from $\tilde f_0$:

$$\sup_{\tilde v \in \tilde V} \|\tilde f_0(\tilde v) - \tilde f(\tilde v)\| \leqslant \text{const} < \infty.$$

This is a corollary of the universal (compressing) property of l^∞-norms (see §§2.1, 4.1). The map $\tilde{f}$ is not necessarily invariant under the actions of the group $\mathbf{Z}^r = H_1(V; \mathbf{Z})$ in the spaces $\tilde{V}$ and $H_1(V; \mathbf{R})$. However, one may average $\tilde{f}$ over the group $\mathbf{Z}^r$ to a $\mathbf{Z}^r$-*invariant* distance-decreasing map $\tilde{V} \to H_1(V; \mathbf{R})$. Thus one gets a map $f\colon V \to J_1$ homotopic to f_0 and distance-decreasing relative to the (flat Finsler) metric in J_1 induced by the l^∞-norm in $H_1(V; \mathbf{R})$.

We are interested in a mass*-extremal norm in $H_1(V; \mathbf{R})$, which is $\leqslant \mathrm{Vol}_{\mathbf{R}}$. We approximate this extremal norm by strictly smaller l^∞-norms, and get in the limit a map $f\colon V \to J_1$ homotopic to f_0 which is distance-decreasing relative to this extremal norm. In particular, the map f is mass*-decreasing on all r-dimensional submanifolds of V.

If the map f_0 has degree $d > 0$, then we get (for the second time) the inequality

$$\mathrm{mass}^* J_1 \leqslant d^{-1} \mathrm{mass}^* V.$$

Now let $r < n$, and take the pullback $\tilde{\Delta}$ of a generic point $x \in H_1(V; \mathbf{R})$ under the map $\tilde{f}_0$:

$$\tilde{\Delta} = \tilde{f}_0^{-1}(x) \subset \tilde{V}.$$

This $\tilde{\Delta}$ is an oriented (V is supposed to be oriented) submanfold in $\tilde{V}$ of dimension $n - r$, whose homology class $[\tilde{\Delta}] \in H_{n-r}(\tilde{V}; \mathbf{Z})$ is a homotopy invariant of the manifold V. We generalize the above nondegeneracy condition to the case $n > r$ by requiring the class $[\tilde{\Delta}]$ to be nontrivial, and denote by $\deg = \deg(V, \text{the metric in } V)$ the lower bound of the masses* of cycles in $\tilde{V}$ homologous to $\tilde{\Delta}$. If V is a Riemannian manifold (rather than a general Finsler manifold), then there exists a subvariety $\tilde{\Delta}_{\min}$ in V of the least volume, which is homologous to $\tilde{\Delta}$, and thus

$$\mathrm{Vol}\, \tilde{\Delta}_{\min} = \deg(V).$$

7.5.B. Theorem. *Let V be a Finsler manifold, for which* $r = \mathrm{rank}\, H_1(V; \mathbf{R}) \leqslant n = \dim V$. *Then the mass* of the Jacobian $J_1 = J_1(V)$ satisfies*

$$\mathrm{mass}^* J_1 \leqslant (\deg)^{-1} \mathrm{mass}^* V.$$

Proof. Apply the coarea formula to the mass*-decreasing map $f\colon V \to J_1$.

Example. Let V be a closed aspherical 3-dimensional Finsler manifold whose fundamental group is nilpotent with generators a, b, c and relations $[a, c] = [b, c] = 1$ and $[a, b] = c$. (The element c generates the center of $\pi_1(V)$.) Denote by γ the shortest geodesic in V, which is homotopic to some power c^k for $k \neq 0$.

The homology group $H_1(V; \mathbf{Z})$ is freely generated by the classes $[a]$ and $[b]$, while the class $[\tilde{\Delta}] \subset \tilde{V} \approx S^1 \times \mathbf{R}^2$ is homologous to the lift of c to $\tilde{V}$.

Therefore

$$\text{mass}^* J_1 \leqslant (\text{length}\,\gamma)^{-1}\,\text{mass}^* V.$$

This inequality becomes more interesting with the following geometric interpretation of $\text{mass}^* J_1$ in terms of the volumes of the balls $B(R)$ in the *universal* covering $\tilde{\tilde{V}}$ of V.

7.5.C. Theorem (*Pansu* [62]). *If $R \to \infty$, then*

$$R^{-4}\,\text{mass}^* B(R) \to \nu^*\,\text{mass}^* V/(\text{mass}^* J_1)^2,$$

where ν^* is a certain geometric invariant of the $\text{Vol}_{\mathbf{R}}$*-norm in the space* $H_1(V; \mathbf{R}) \approx \mathbf{R}^2$, *which satisfies* $0.1 \leqslant \nu^* \leqslant 10$.

Furthermore, suppose that the minimal geodesic $\gamma \sim c^k$ in V (which is non-contractible but yet homologous to zero in V) has

$$\text{Fill Vol}(\gamma \subset V) \leqslant \text{Is}_1\,\text{length}\,\gamma$$

for some constant $\text{Is}_1 = \text{Is}_1(V)$. Then (compare the "area" inequality of §7.4.C′)

$$\text{Is}_1\,\text{length}\,\gamma \geqslant \text{mass}^* J_1,$$

so

$$[\text{mass}^* J_1]^2 \leqslant \text{Is}_1\,\text{mass}^* V, \quad \lim_{R\to\infty} R^{-4}\,\text{mass}^* B(R) \geqslant 0.1/\text{Is}_1.$$

Let us prove an analogous relation with the *first nonzero eigenvalue* $\lambda_1 = \lambda_1(V)$ of the Hodge-Laplace operator on 1-*forms on* V, assuming V is a *Riemannian* manifold. If ω is an exact 2-form on V, then by the Hodge theory there exists a 1-form l on V such that

$$\omega = dl,\ \|l\|_{L_2} \leqslant \lambda_1^{-1/2}\|\omega\|_{L_2}.$$

Consider the above Abels' map $f\colon V \to J_1$, and let ω be the pullback of the normalized area form ω_0 on J_1:

$$\omega = f^*(\omega_0), \quad \text{for} \int_{J_1} \omega_0 = 1.$$

The form ω is exact, and the equation $dl = \omega$ implies $|\int_V \omega \wedge l| = 1$.

Now if f is a mass*-decreasing map, then $\|\omega\|_v \leqslant (\text{mass}^* J_1)^{-1}$ for all $v \in V$, and so $\|\omega\|_{L_2} \leqslant (\text{Vol}\,V)^{1/2}/\text{mass}^* J_1$. If we take a one-form l, for which $dl = \omega$ and $\|l\|_{L_2} \leqslant \lambda_1^{1/2}\|\omega\|_{L_2}$, then

$$1 = \left|\int_V \omega \wedge l\right| \leqslant \|\omega\|_{L_2}\|l\|_{L_2} \leqslant \lambda_1^{-1/2}(\text{Vol}\,V)/(\text{mass}^* J_1)^2,$$

and so

$$\lim_{R\to\infty} R^{-4}\,\text{Vol}\,B(R) \geqslant 0.1\lambda_1^{1/2}.$$

8. Visual hulls and minimal subvarieties in Riemannian manifolds

Our filling results in §§4–6 are based on a very coarse solution of the Plateau problem in some *Finsler* spaces. By the available technique of the geometric measure theory one can obtain much sharper results for *Riemannian* manifolds.

8.1. Visual Volume. Denote by E_x the radial projection of the space $\mathbf{R}^N$ onto the unit sphere S_x^{N-1} around a point $x \in \mathbf{R}^N$. Then we apply this projection E_x to an n-dimensional, for $n < N$, submanifold V in $\mathbf{R}^N$, and denote by $\mathrm{Jac}(v; x)$ the (absolute value of) Jacobian of E_x at a point $v \in V$ for $v \neq x$. We introduce the *visual volume* of V from x as the normalized total n-dimensional volume (counted with geometric multiplicity) of the map E_x: $V \to S_x^{N-1}$; namely,

$$\mathrm{Vis}(V; x) = \int_V \mathrm{Jac}(v; x)\, dv / \mathrm{Vol}\, S^n,$$

for the unit sphere $S^n \subset \mathbf{R}^{n-1}$.

This definition generalizes to submanifolds V in an arbitrary complete simply connected Riemannian manifold X *without conjugate points* and also to submanifolds in the standard sphere S^N; namely, one takes the radial geodesic projection of X onto the unit tangent sphere $S_x^{N-1} \subset T_x(X)$ for the map E_x.

The visual volume $\mathrm{Vis}(V; x)$ is apriori defined for the points x outside V (and outside the symmetric image of V in case $X = S^N$). However, if V is a smooth (immersed or embedded) submanifold in X of positive codimension, then the function $\mathrm{Vis}(V; x)$ extends continuously to all interior points of the manifold V. In particular, if V is a manifold without boundary, and the embedding (immersion) $V \to X$ is *proper*, then the function $\mathrm{Vis}(V; x)$ is continuous on X unless it is everywhere $= \infty$.

Examples. (1) For a linear subspace $\mathbf{R}^n \subset \mathbf{R}^N$, the visual volume is identically $1/2$,

$$\mathrm{Vis}(\mathbf{R}^n; x) = 1/2, \quad \text{for all } x \in \mathbf{R}^N.$$

(2) For an arbitrary submanifold $V \subset X$ and every *interior* point v in V,

$$\mathrm{Vis}(V; v) \geqslant 1/2,$$

where the equality may hold only if V is contained in an n-dimensional totally geodesic submanifold of X through the point v.

(3) If the manifold X has constant sectional curvature, and the submanifold $V \subset X$ meets every totally geodesic submanifold in X of dimension $= (\dim X - \dim V)$ at at most d points, then

$$\mathrm{Vis}(V; x) \leqslant d/2,$$

for all $x \in X$. This happens, for instance, when V is a *real algebraic* subvariety in $\mathbf{R}^N$ of degree $\leqslant d$.

(4) For a *closed* submanifold $V \subset X$ and every point $v \in V$,

$$\operatorname{Vis}(V; v) \geqslant 1,$$

where the equality may hold only for boundaries of geodesically convex $(n+1)$-dimensional submanifolds $W \subset X$. In this case $\operatorname{Vis}(V, x) = 1$ for all $x \in W$, and also $\operatorname{Vis}(V, x) < 1$ for all points $x \in X$ outside W unless $X = S^N$.

(5) For an arbitrary properly immersed submanifold V in $\mathbf{R}^N$ of finite volume,

$$\operatorname{Vis}(V; v) \geqslant 1, \quad \text{for all } v \in V.$$

This is also true if V has sub-Euclidean growth, that is, if the intersections of V with concentric balls in $\mathbf{R}^n$ of radius $R \to \infty$ satisfy

$$R^{-n} \operatorname{Vol}(V \cap B(R)) \to 0.$$

(6) For an arbitrary submanifold V in $\mathbf{R}^N$,

$$\operatorname{Vis}(V; x) < \sigma_n^{-1} [\operatorname{dist}(V, x)]^{-n} \operatorname{Vol} V,$$

where σ_n is the volume of the unit sphere $S^n \subset \mathbf{R}^{n+1}$. The equality holds only for round n-spheres with center x.

(7) If the manifold X has nonpositive sectional curvature, and V is a flat totally geodesic submanifold in X, then

$$\operatorname{Vis}(V; x) \leqslant 1/2, \quad \text{for all } x \in X.$$

(8) Let V_0 be a closed submanifold in X of codimension $\geqslant 2$, and V_ε the boundary of a small ε-neighborhood $U_\varepsilon(V) \subset X$. If $\varepsilon \to 0$, then the functions $\operatorname{Vis}(V_\varepsilon; x)$ pointwise converge to the characteristic function of V_0, which is one on V_0 and zero outside V_0. This convergence is uniform outside any given neighborhood $U_\varepsilon(V_0)$ for $\varepsilon > 0$, and the supremums over x of the functions $\operatorname{Vis}(V_\varepsilon; x)$ converge to one. The topology of the manifolds V in this example does not give any nontrivial lower bound for the supremum (over x) of the function "Vis". However, there are some relations between the topology of manifolds V in $\mathbf{R}^N$ and $\sup_{x \in \mathbf{R}^N} \operatorname{Vis}(V; x)$. For instance, nontrivial knots in $\mathbf{R}^3$ always have this supremum > 2 (see Examples 8.2.B below).

Following the ideas of Paul Levy (see [53, Part III, Chap. V]) we introduce the following.

8.1.A. Definition. *The visual hull* of a submanifold V in X is a closed subset,

$$\operatorname{Vi\,Hull}(V) \subset X,$$

of those points $x \in X$, for which $\operatorname{Vis}(V; x) > 1$.

Examples. The visual hull of a closed connected *curve* V in $\mathbf{R}^N$ always contains the *convex hull* of V, as any hyperplane in $\mathbf{R}^N$, which intersects the convex hull, meets the curve at least twice. In fact, the visual hull is strictly greater than the convex hull unless the curve is a plane convex curve.

The visual hulls of the above manifolds V_ε do not necessarily contain the respective convex hulls. In fact, if $\varepsilon \to 0$, these visual hulls converge to the manifold V_0.

8.1.B. Theorem. *Every closed submanifold $V \subset X$ bounds inside the visual hull, that is, the inclusion map $V \to Vi\,\mathrm{Hull}(V)$ sends the fundamental class $[V] \in H_n(V)$, $n = \dim V$, to zero.*

Proof. Take a submanifold Y in the complement $X \backslash V$ of $\dim Y = N - n - 1$ for $N = \dim X$, and consider the family of maps $E_y \colon V \to S_y^{N-1}$ to the tangent spheres at the points $y \in Y \subset X$. This family defines a map E of the product $V \times Y$ to the unit sphere $S^{N-1} \subset \mathbf{R}^N$, and the degree $\deg E$ equals the *linking number* between the manifolds V and Y.

If V does not bound inside some larger set $U \supset V$ in X (the relevant U is the visual hull of V), then there is a submanifold (or at least a pseudomanifold) Y in X outside U such that

$$\text{linking number}\,(V, Y) \neq 0.$$

Thus the theorem is reduced to

8.1.B′. Proposition. *The condition $\deg E \neq 0$ implies the existence of a point $y \in Y$ for which the volume of the map E_y (that is, the integrated Jacobian of this map) satisfies*

$$\operatorname{Vol} E_y \geqslant \operatorname{Vol} S^n,$$

for the unit sphere $S^n \subset \mathbf{R}^{n+1}$.

Proof. For $\dim V = 1$ the proof is trivial, as every curve $E_y(V)$ in S^{N-1} of length $< 2\pi$ is contained in an open hemisphere and therefore can be *canonically* contracted to some point in this hemisphere. This shows that the map E is contractible, and so $\deg E = 0$.

Next we look at another simple case; namely, we assume $\dim V = N - 2$, and suppose for simplicity's sake that the maps E_y are imbeddings. Since $\deg E \neq 0$, there is a point $y_0 \in Y$ for which the image of the map E_{y_0}: $V \to S^{N-1}$ divides the range sphere S^{N-1} into two parts of *equal volumes*. Then the classical isoperimetric inequality for S^{N-1} implies

$$\operatorname{Vol} E_{y_0} \geqslant \operatorname{Vol} S^{N-2}.$$

Observe that this proof works with minor modifications for maps E_y which are not necessarily imbeddings, but yet only for $\dim V = N - 2$.

Now the general case of the proposition reduces to the following facts.

Fact 1. *Let V and Y be closed manifolds (or pseudomanifolds), and E a continuous map of the product $V \times Y$ to a closed Riemannian manifold S. If* $\deg E \neq 0$, *then there exists an n-dimensional, $n = \dim V$, minimal subvariety M in S, whose volume is not greater than the volumes of the restricted maps E_y: $V \to S$ for $E_y = E \mid V \times y = y$ and all $y \in Y$.*

This is a corollary of the Almgren-Morse theory (see [5]).

Fact 2. *An arbitrary n-dimensional minimal subvariety M in the unit sphere S^{N-1} has*

$$\operatorname{Vol} M \geqslant \operatorname{Vol} S^n.$$

See [52].

Unfortunately, both Facts have long and difficult proofs, as they depend on the regularity theorems of Almgren and Allard (see [6], [2]). A direct elementary proof of Proposition 8.1.B′ is yet to be found.

Theorem 8.1.B generalizes the following result of Bombieri and Simon, which is the solution of *Gehring's linking problem*. (See [16].)

8.1.C. Theorem. *The filling radius of a closed n-dimensional submanifold $V \subset \mathbf{R}^N$ admits the following (sharp!) upper bound*

$$\operatorname{Fill\,Rad}(V \subset \mathbf{R}^N) \leqslant [\operatorname{Vol}(V)/\operatorname{Vol} S^n]^{1/n},$$

with the equality for round n-spheres in $\mathbf{R}^{n+1} \subset \mathbf{R}^N$.

Proof. Theorem 8.1.B implies Theorem 8.1.C, as the visual hull of V is contained in the ε-neighborhoods of V for $\varepsilon = [\operatorname{Vol}(V)/\operatorname{Vol} S^n]^{1/n}$ where S^n is the unit sphere.

8.2. Singularities of minimal varieties and the visual volume. Bombieri and Simon have proved their theorem by analyzing a *minimal* filling W of V. A slight modification of their method provides additional geometric information concerning minimal fillings of V.

Recall that the (upper) *density* of an $(n+1)$-dimensional subvariety W in $\mathbf{R}^N$ at a point $v \in W$ is

$$\limsup_{\rho \to 0} \left[\operatorname{Vol}\left(W \cap B_w^N(\rho)\right)/\operatorname{Vol} B^{n+1}(\rho)\right],$$

where $B_w^N(\rho)$ is the Euclidean ball at w of radius ρ, and $B^{n+1}(\rho)$ is the ρ-ball in $\mathbf{R}^{n+1}$.

If W is a *minimal* subvariety in $\mathbf{R}^N$ with boundary, then all interior points have density at least one:

$$\operatorname{Dens}_w(W) \geqslant 1,$$

and the boundary points have

$$\operatorname{Dens}_w(W) \geqslant 1/2, \ w \in \partial W.$$

Furthermore, the *singular* points in the interior have density $\geqslant 1 + \varepsilon$ for some universal positive number $\varepsilon = \varepsilon(N) > 0$, and the boundary singular points have density $\geqslant 1/2 + \varepsilon$. These are deep theorems of Allard (see [2], [52]).

8.2.A. Theorem. *Let V be a smooth n-dimensional submanifold in $\mathbf{R}^N$, and W a minimal $(n+1)$-dimensional variety with boundary $\partial W = V$. Then for every interior point $w \in W$,*

$$\operatorname{Dens}_w(W) \leqslant \operatorname{Vis}(V, w), \tag{8.1}$$

and for every boundary point $v \in V$,

$$\operatorname{Dens}_v(W) \leqslant \operatorname{Vis}(V, v) - \tfrac{1}{2}. \tag{8.2}$$

Remark. As any submanifold V bounds at least one minimal variety (for example, the variety of least volume), Theorem 8.2.A indeed sharpens Theorem 8.1.B for $X = \mathbf{R}^N$.

Proof. We prove (8.1) and (8.2) by applying the first variation formula to the *radial* field $Z = Z_w$ in $\mathbf{R}^N$. The word "radial" means that Z is invariant under rotations of $\mathbf{R}^N$ around the fixed point w. The field Z which we need is a product of the standard field $x - w$, for $x \in \mathbf{R}^N$, by a positive function such that the following two conditions are satisfied:

(a) The restriction of Z to any linear subspace $\mathbf{R}^{n+1} \subset \mathbf{R}^N$ through w has zero divergence.

(b) The flux of Z through any n-sphere in every $\mathbf{R}^{n+1}_w$ around the point w equals one.

Observe that the radial field Z which satisfies (a) and (b) is unique. This Z on every subspace $\mathbf{R}^{n+1}_w$ equals the gradient of the fundamental solution of the Laplace operator on $\mathbf{R}^{n+1}_w$, and so the field Z has a singularity at the point w.

Observe furthermore that the *divergence* of Z on every $(n+1)$-dimensional submanifold $W \subset \mathbf{R}^{n+1}$ is nonnegative; the volume of W may only increase under the flow in $\mathbf{R}^N$ generated by Z. Moreover, the divergence of Z on the cone from w over the submanifold $V \subset \mathbf{R}^N$ is zero. The flux of Z through V in this cone is exactly the visual volume $\operatorname{Vis}(V, w)$, provided that the radial projection $E_w\colon V \to S^{N-1}_w$ is almost everywhere injective. Otherwise, this flux may be only *less* than the visual volume.

Take the normal projection $\bar{Z}$ of the field Z on the *minimal* variety W at all regular points of W. This new field $\bar{Z}$ has nonnegative divergence on W (it expands the volume of W), while the flux of $\bar{Z}$ through V is not greater than the corresponding flux of Z in the cone, which in turn is not greater than $\operatorname{Vis}(V, w)$. (Compare [16], [52].) Therefore the visual volume may be only greater than the flux of $\bar{Z}$ through intersections of W with infinitesimally small spheres $S^{N-1}_w(\varepsilon)$, $\varepsilon \to 0$. The limit of the latter flux as $\varepsilon \to 0$ equals the density

$\mathrm{Dens}_w(W)$, and thus the proof is complete for interior points of W. The analysis of boundary points is essentially the same.

We refer to [52] for basic properties of minimal varieties, which are necessary to justify the above arguments.

Observe that Bombieri and Simon use another radial field to estimate a lower bound of the *volume* of V, rather than the visual volume.

8.2.B. Examples. By the results of Allard mentioned above, Theorem 8.2.A implies that there are no singularities on minimal varieties W in $\mathbf{R}^N$, whose boundary $W = V$ satisfies

$$\mathrm{Vis}(V; x) \leqslant 1 + \varepsilon, \tag{8.3}$$

for all $x \in \mathbf{R}^N$ and some universal number $\varepsilon = \varepsilon(N) > 0$. In particular, any *closed manifold* V in $\mathbf{R}^N$ *which satisfies* (8.3) *bounds a smooth manifold, and so all characteristic numbers of* V *vanish.*

If W is a parametrized minimal surface ($\dim W = 2$), then the only possible singularities are double points and branched points. Therefore one may take $\varepsilon = 1$. In particular, a *simple closed curve* V in $\mathbf{R}^3$, *for which* $\mathrm{Vis}(V, x) \leqslant 2$ *for all* $x \in \mathbf{R}^3$, *bounds an imbedded minimal disk, and so the curve is unknotted.*

This result is sharp. If the trefoil knot V below converges to the doubly covered circle, then $\sup_{x \in \mathbf{R}^3} \mathrm{Vis}(V; x) \to 2$.

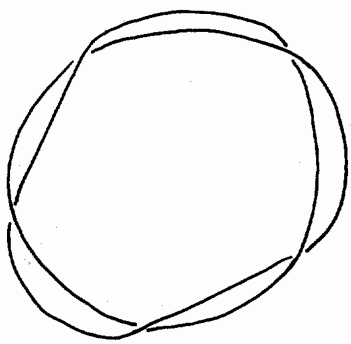

FIG. 2

The author does not know any elementary proof of the above facts, and whether the inequality $\sup_x \mathrm{Vis}(V; x) \leqslant \text{const}$ for a large "const" imposes any topological restrictions on V. Such restriction does not appear for simply connected submanifolds of high codimension. In fact, every closed simply connected manifold V of $\dim V \geqslant 4$ can be obtained by a surgery in codimensions $\geqslant 2$ from a disjoint union of some standard manifolds (compare [38]). A natural geometric realization of this surgery in the space $\mathbf{R}^N$, for any given

$N \geqslant 2 \dim V$, imbedds V into $\mathbf{R}^N$ with sup $\operatorname{Vis}(V; x) \leqslant$ const for some universal constant const $=$ const(n). This argument also yields *open* properly imbedded submanifolds V in $\mathbf{R}^N$ of *finite volume*, for which $\sup_x \operatorname{Vis}(V; x) \leqslant$ const $=$ const(n).

8.3. Minimal varieties in the hyperbolic space H^N. Theorem 8.2.A generalizes to the hyperbolic space (of constant curvature -1) and also to subvarieties in a hemisphere in S^N. Furthermore, let V be a smooth closed n-dimensional submanifold in the *ideal boundary* $\partial H^N = S^{N-1}$ of the hyperbolic space H^N. Then one may consider complete minimal varieties W in H^N, whose *ideal boundary* $\partial W \subset \partial H^N$ equals V; namely, one takes a geodesic cone from a point $x \in H^N$ over $V \subset \partial H$, and requires W to be asymptotic to this cone.

To understand these minimal varieties we take the convex hull of V in H^N,

$$\operatorname{Conv Hull}(V) \subset H^N,$$

and observe that this hull is contained in some ε-neighborhood (for $\varepsilon < \infty$!) of any (asymptotic) geodesic cone over V. In fact, this hull exponentially approaches the cone at infinity as it is clearly seen in the *projective model* of H^N. The convex hull of V even has *finite* volume for $N \geqslant 2 \dim V + 2$.

The radial projection on the unit tangent spheres E_x: $H^N \to S_x^{N-1}$, for $x \in H^N$, extends to the boundary $S^{N-1} = \partial H^N$, and so one has the visual volume $\operatorname{Vis}(V; x)$, $x \in H^N$, defined for submanifolds $V \subset S^{N-1} = \partial H^N$. The function $\operatorname{Vis}(V; x)$ is an eigenfunction (in x) of the Laplace operator on H^N, and if points x *in the convex hull* of V approach infinity, then $\operatorname{Vis}(V; x) \to 1$ (see [72]).

We say that a complete (i.e., closed in H^N as a subset) subvariety W in H^N *spans* V in ∂H^N, if the closure of the projective image of W in the unit ball B^N spans the image of V in $S^{N-1} = \partial B^N$, that is, the fundamental class of V vanishes in the (spectral) homology of this closure.

If such a spanning subvariety W in H^N is a *minimal* variety of dimension $n + 1$, then it is contained in the convex hull $\operatorname{Conv Hull}(V) \subset H^N$. Every such W probably satisfies inequality (8.1). We shall now prove this inequality for a special class of minimal varieties W.

Take the intersection of W with the ball of radius R around some point $w \in W$, and divide the volume of this intersection by the volume of the hyperbolic $(n + 1)$-ball. Denote this ratio by

$$\operatorname{Dens}_w(W; R) = \operatorname{Vol}\big(W \cap B_w^N(R)\big)/\operatorname{Vol} B_H^{n+1}(R).$$

If W is a minimal variety, then the function $\operatorname{Dens}_w(W; R)$ is monotone nondecreasing in R. (See [52].)

Now we require W to satisfy the following two conditions:

$$\text{(1)} \qquad \liminf_{R\to\infty} R^{-1} \log \operatorname{Dens}_w(W; R) < \infty$$

for some point $w \in W$. This is equivalent to

$$\text{(8.4)} \qquad \liminf_{R\to\infty} R^{-1} \log \operatorname{Vol}\big(W \cap B_w^N(R)\big) < \infty,$$

for all points $w \in W$.

(2) W is asymptotically volume-minimizing. That is, for every point $w \in W$ and every $\varepsilon > 0$ there is a number $R_0 > 0$ such that the intersection $W \cap B_w^N(R)$, for every $R > R_0$, has a volume smaller than $1 + \varepsilon$ times the volume of any other variety which spans the boundary of the intersection $\partial(W \cap B_w^N(R) = W \cap S_w^{N-1}(R)$.

(Probably, all minimal varieties which span V satisfy (1) and (2).)

Example (*Anderson*). Let us project the manifold $V \subset \partial H^N$ on the sphere $S_w^{N-1}(R) \subset H_N$ around a point $w \in \operatorname{Conf Hull} V \subset H^N$, and let us span this projection by a volume minimizing variety $W(R)$ in the ball $B_w^N(R) \subset H^N$. We compare this variety $W(R)$ with the geodesic cone from the point w over V and get

$$\operatorname{Dens}_w(W(R); R) \leqslant \operatorname{Dens}_w(\text{Cone}) = \operatorname{Vis}(V; w).$$

Therefore for every point $w' \in W(R)$ within distance ρ from w and for every $R' \leqslant R - \rho$, we get

$$\operatorname{Dens}_{w'}(W(R); R') \leqslant \exp(n\rho)\operatorname{Vis}(V; w).$$

This gives a uniform bound for the volumes of the intersections $W(R) \cap B_{w'}^N(R')$ for any fixed ball $B_{w'}^N(R')$ in H^N and for $R \to \infty$. Thus some subsequence of the varieties $W(R)$, $R \to \infty$, converges (in the flat topology) to a minimal variety $W \subset \operatorname{Conv Hull}(V) \subset H^N$, which spans $V \subset \partial H^N$ and satisfies the conditions (1) and (2).

Now we claim the following version of Theorem 8.2.A.

8.3.A. Theorem. *If a minimal variety W in H^N spans a manifold $V \subset \partial H^N$, and satisfies the above conditions* (1) *and* (2), *then*

$$\operatorname{Dens}_w(W; R) \leqslant \operatorname{Vis}(V; w),$$

for all points $w \in W$ and all real numbers $R > 0$.

Proof. Denote by $V(R) \subset S_w^{N-1}(R)$ the projection of V on the sphere $S_w^{N-1}(R) \subset H^N$, and let us replace the intersection of W with the ball $B_w^n(R)$ by the union of the cone over $V(R)$ from w and the cylinder of the normal projection of the boundary $\partial(W \cap B_w^N(R)) = W \cap S_w^N(R)$ on the manifold $V(R)$. This operation gives a new variety with the same boundary as the intersection $W \cap B_w^N(R)$, and according to (2) the volume of this new variety may be only slightly smaller than the volume of $W \cap B_w^N(R)$ for large R. As the convex hull of V exponentially narrows around the cone over V for $R \to \infty$, we can estimate the volume of the new variety by (the volume of the cone) + (an exponentially small factor) $\times$ (the n-dimensional volume of the boundary of $W \cap B^N(R)$). Next we use the coarea inequality,

$$\mathrm{Vol}(W \cap B_w^N(R) \leqslant \int_0^R \mathrm{Vol}(W \cap S_w^{N-1}(\rho)\, d\rho,$$

and then the property (1) implies

$$\lim_{R \to \infty} \mathrm{Dens}_w(W, R) \leqslant \mathrm{Vis}(V, w).$$

Remarks. (a) Using the normal projection of W on the cone over V one can see that

$$\lim_{R \to \infty} \mathrm{Dens}_w(W, R) = \mathrm{Vis}(V, w),$$

for all $w \in W$.

(b) The apriori estimates of Allard [2] seem to imply not only the absence of singularities of W at the points $w \in W$, for which the density

$$\mathrm{Dens}_w(W) = \lim_{R \to 0} \mathrm{Dens}_w(W, R)$$

is close to one, but also a universal bound on the extrinsic curvature of W at the points w, for which the density $\mathrm{Dens}_w(W, 1)$ is close to one. If such universal bound exists, then the extrinsic curvature in our case must be necessarily close to zero, and thus the induced curvature in W is close to -1. This would make W homeomorphic to $\mathbf{R}^{n+1}$, provided $\pi_1(V) = 0$ and

$$\sup_{x \in H^N} \mathrm{Vis}(V; x) \leqslant 1 + \varepsilon,$$

for some universal $\varepsilon = \varepsilon(N) > 0$. In fact, by such a universal bound for the extrinsic curvature one can prove that the closure of the projective image of W is a smooth ball with $\partial W = V$, provided that $\sup_x \mathrm{Vis}(V; x) \leqslant 1 + \varepsilon$, and that there is no assumption on the fundamental group $\pi_1(V)$. In particular, the manifolds V in $S^{N-1} = \partial H^N$ with small visual volumes would be (proven to be) smooth spheres.

In a special case, namely for circles in S^3, one does not need any apriori estimates to get W diffeomorphic to a disk, as one always can span the circles

$$V(R) \subset S_w^3(R) = \partial B_w^4(R) \subset H^4$$

by disks. These disks are nonsingular for large R if $\sup_x \mathrm{Vis}(V, x) \leqslant 2$.

Example. The trefoil knot V in S^3 can not be spanned by any embedded disk in B^4, and so the visual volume of this V from some points $w \in H^4$ is greater than 2. One can express this property of the trefoil knot $V \subset S^3$ by using the conformal geometry of the unit sphere S^3. Namely there is a conformal transformation of S^3 which makes the length of V greater than 4π. (Compare [54].)

Questions. What is the asymptotic behavior of minimal varieties in spaces X of nonpositive curvature? It seems that the asymptotic Plateau's problem is solvable in manifolds X, whose sectional curvature is pinched between -1 and -4, if one takes for $V \subset \partial X$ the radial projection to infinity of a smooth submanifold in a tangent sphere $S_x \subset T_x(X)$. What is the situation in symmetric spaces X? See Berkeley's thesis (1981) by M. Anderson for beautiful results in this direction.

8.4. On the volume of visual hulls. Let V be an n-dimensional submanifold in $\mathbf{R}^N$, and let $U \subset \mathbf{R}^N$ be an arbitrary measurable subset of the total N-dimensional measure μ. Let us estimate the integral

$$\int_U \operatorname{Vis}(V; u)\, du.$$

For a fixed point $v \in V$ we treat the Jacobian of the map $E_x\colon V \to S_x^{N-1}$ as the function of x,

$$J(x) = \operatorname{Jac}(v; x)$$

There is a unique number $t > 0$ for which the pullback $J^{-1}[t, \infty)$ satisfies

$$\operatorname{Vol} J^{-1}(t) = \mu = \operatorname{Vol} U.$$

Denote this pullback by $U' = U'(v, \mu) \subset \mathbf{R}$ and observe that

$$I_v = \int_U J(u)\, du \leqslant \int_{U'} J(u')\, du' = I'_v.$$

The integral I' is, in fact, a unique function of μ, which can be explicitly calculated. We need only the following crude estimate for I'_v by the integral of the function $[\operatorname{dist}(v, x)]^{-n} = \delta(x) \geqslant J(x)$ over the ball $B = B_v^N(R)$ for which $\operatorname{Vol} B = \mu = \operatorname{Vol} U' = \operatorname{Vol} U$:

$$(8.5) \qquad I'_v \leqslant \int_B \delta(x)\, dx = (N-n)^{-1}(\operatorname{Vol} S^{N-1}) R^{N-n} = C' \mu^{(N-n/N)},$$

$$\text{for } C' = N^{(N-n)/n}(N-n)^{-1}(\operatorname{Vol} S^{N-1})^{(2N-n)/N}.$$

Then

$$(8.6) \qquad \int_U \operatorname{Vis}(V; u)\, du = (\operatorname{Vol} S^n)^{-1} \int_V I_v\, dv \leqslant C'(\operatorname{Vol} S^n)^{-1} \mu^{(N-n)/N} \operatorname{Vol} V.$$

We apply (8.6) to $U = \text{Vi Hull}(V)$. We have, by definition,

$$\text{Vis}(V, u) \geqslant 1, \quad \text{for } u \in U = \text{Vi Hull}(V),$$

and therefore

$$\int_U \text{Vis}(V; u)\, du \geqslant \text{Vol}\, U = \mu.$$

Thus we obtain the following *estimate for the volume* $\mu = \text{Vol}(\text{Vi Hull}\, V)$:

$$\mu \leqslant C''(\text{Vol}\, V)^{N/n}, \tag{8.7}$$

for

$$C'' = (C')^{N/n}(\text{Vol}\, S^n)^{-N/n}$$
$$= N^{N(N-n)/n^2}(N-n)^{-N/n}(\text{Vol}\, S^{N-1})^{(2N-n)/n}(\text{Vol}\, S^n)^{-N/n}.$$

The visual hull of a closed *hypersurface* V in $\mathbf{R}^{n+1}$ contains the region $W \subset \mathbf{R}^{n-1}$ bounded by V, and then (8.7) yields the ordinary isoperimetric inequality

$$\text{Vol}\, W \leqslant C''(\text{Vol}\, V)^{(n+1)/n}.$$

One can improve this constant $C'' = C''_n$ by explicitly evaluating the integral I'_v, but this only makes the inequality sharp for $n = 1$. (See [68], [7], [63] for variations and generalizations of this argument.)

9. Distortion of maps and submanifolds

If one wants to apply the results of the previous section (in particular, the Bombieri-Simon estimate for Fill Rad$(V \subset \mathbf{R}^N)$) to an abstract (not embedded) Riemannian manifold V, one should decide whether V admits a map f into some space $\mathbf{R}^N$ which does not distort the metric very much. Recall that the *dilation* (Lipschitz constant) of f is

$$\text{Dil}\, f \overset{\text{def}}{=} \sup[\text{dist}(f(v_1), f(v_2))/\text{dist}(v_1, v_2)],$$

over all pairs of points v_1 and $v_2 \neq v_1$ in V. The *distortion* of a homeomorphism f of V onto a submanifold in $\mathbf{R}^N$ is defined as the product

$$\text{Distor}(f) = (\text{dil}\, f)(\text{dil}\, f^{-1}).$$

Furthermore, every submanifold $V \subset \mathbf{R}^N$ carries two natural metrics: the first is induced from $\mathbf{R}^N$ and the second corresponds to the *Riemmanian structure* induced from $\mathbf{R}^N$. The distance in the second metric is the lower bound of the lengths of curves in V, which join given points in V. The *distortion* of V is defined as the distortion of the identity map $V \to V$ relative to this pair of metrics.

Examples. (1) Every closed curve in $\mathbf{R}^N$ has distortion $\geqslant \pi/2$, where the equality holds only for circles. (See [37].) This implies the inequality distor$(V) \geqslant \pi/2$ for those compact (possibly with boundary) submanifolds $V \subset \mathbf{R}^N$, which either have $\pi_1(V) \neq 0$ or admit a fixed point free involution preserving the induced Riemannian structure.

(2) Let x, y, and z be orthonormal coordinates in $\mathbf{R}^3$. Let V be the closed surface in $\mathbf{R}^3$, which is the union of the disk $\{x^2 + y^2 = 1, z = 0\}$ and the cone from the point $(0, 0, 10) \in \mathbf{R}^3$ over the circle $\{x^2 + y^2 = 1, z = 0\}$. A straightforward calculation shows that

$$\text{distor}\, V < \pi/2 - 10^{-10}.$$

(3) If a compact submanifold V in $\mathbf{R}^N$ has distortion $\leqslant \pi/2\sqrt{2}$, then V is contractible (see the Appendix by Pansu in [37]). Therefore every map of a non-contractible Riemannian manifold into $\mathbf{R}^N$ has distortion $\geqslant \pi/2\sqrt{2} > 1$.

(4) The argument in [37], when applied to submanifolds V in the *hyperbolic* space H^N, bounds the distortion of $V \subset H^N$ from below roughly by

$$\exp[\text{Fill Rad}(V \subset H^N)].$$

If V' is an abstract Riemannian manifold with *large* filling radius $R' =$ Fill Rad V', then any map $V' \to H^N$ has distortion roughly greater than $R'/\log R'$. It follows that every map $\mathbf{R}^n \to H^N$ has infinite distortion for $n \geqslant 2$.

Conversely, there is no map $H^n \to \mathbf{R}^N$, $n \geqslant 2$, of finite distortion. Indeed, the hyperbolic space H^n has exponential growth, while the Euclidean space $\mathbf{R}^N$ has polynomial growth. It is unclear, however, whether there are maps of finite distortion of the hyperbolic plane H^2 into the infinite dimensional Hilbert space $\mathbf{R}^\infty$.

(5) An n-dimensional manifold of any given topological type can be realized as a submanifold V in $\mathbf{R}^N$ for large N such that

$$\text{distor}\, V \leqslant 100\sqrt{n}\,.$$

This is proven by induction with some triangulation of V.

The dimension N may depend on the topology of V. What happens in low codimensions is not clear.

Question. Does *every* isotopy class of knots in $\mathbf{R}^3$ have a representative V in $\mathbf{R}^3$ with distortion < 100? Is it so for all torus knots $T_{p,q}$ for $p, q \to \infty$?

9.1. Distortion and spectrum. The geometry of a Riemannian manifold V may impose stronger restrictions on the distortion of maps $V \to \mathbf{R}^N$ than the topology of V. In particular, the first *eigenvalue* of the Laplace operator (on functions) on V, called $\lambda_1 = \lambda_1(V)$, controls the distortion of maps $f\colon V \to \mathbf{R}^N$ as follows. Denote

$$\operatorname{Int} d^2 = \iint_{V\times V} [\operatorname{dist}(v_1, v_2)]^2\, dv_1, dv_2,$$

for the Riemannian distance function on V.

9.1.A. Proposition. *The distortion of an arbitrary map f of a closed n-dimensional Riemannian manifold V into $\mathbf{R}^N$ is bounded from below as follows:*

$$[\operatorname{distor}(f)]^2 \geqslant [\lambda_1(V)\operatorname{Int} d^2]/[2n(\operatorname{Vol} V)^2]. \tag{9.1}$$

Proof. We assume without the loss of generality the map f to be distance-decreasing and then estimate an upper bound of the *square average dilation* of f, which is

$$\iint \|f(v_1) - f(v_2)\|^2\, dv_1\, dv_2 \geqslant [\operatorname{distor}(f)]^{-2} \operatorname{Int} d^2. \tag{9.2}$$

We denote the coordinate functions $V \to \mathbf{R}$ of f by f_i, $i = 1,\cdots,N$, and further assume that $\int_V f_i(v)\, dv = 0$ for $= 1,\cdots,N$. Then

$$\iint \|f(v_1) - f(v_2)\|^2 = 2 \operatorname{Vol} V \sum_{i=1}^{N} \int_V f_i^2(v)\, dv$$

$$\leqslant \lambda_1^{-1} 2 \operatorname{Vol} V \sum_{i=1}^{N} \int_V \|\operatorname{grad} f_i(v)\|^2\, dv.$$

As the map f has $\operatorname{Dil} f \leqslant 1$, we get

$$\sum_{i=1}^{N} \|\operatorname{grad} f_i(v)\|^2 \leqslant n = \dim V,$$

and so

$$\iint \|f(v_1) - f(v_2)\|^2 \leqslant 2n\lambda_1^{-1}(\operatorname{Vol} V)^2.$$

Combining this inequality with (9.2) gives (9.1).

Let us indicate a specific example of a manifold V, for which the right-hand side of (9.1) becomes arbitrary large. We start with a combinatorial argument which is closely related to the "type-cotype" considerations in the geometric theory of Banach spaces (see [70]). These ideas were explained to the author by V. Milman.

Take the unit cube in $\mathbf{R}^d$, and denote the zero and the one-dimensional skeletons of this cube by K^0 and $K^1 \supset K^0$ respectively. Notice that K^0 consists of 2^d vertices and K^1 had $d2^{d-1}$ edges. We define a "Riemannian" metric in K^1 by taking the lengths of the shortest curves in K^1 between pairs of points. Then

$$\text{Diameter}(K^1) = \text{Diameter}(K^0) = d,$$

that is, the square of the Euclidean diameter $= \sqrt{d}$ of the unit cube.

We want to evaluate the combinatorial first eigenvalue of K^0,

$$\lambda_1 = \lambda_1(K^0) \overset{\text{def}}{=} \inf\left[\sum_{K^1} (f(x) - f(y))^2 / \sum_{K^0} f(x)^2 \right],$$

where the first sum is taken over the pairs of points x and y in K^0 with $\text{dist}(x, y) = 1$, and the infimum over the functions $f(x)$, $x \in K^0$, satisfying $\Sigma_{K^0} f(x) = 0$.

For a function $f = f(x)$, let

$$\sum_p^d = \sum (f(x) - f(y))^2, \quad p = 1, \cdots, d,$$

denote the sum of the pairs of points in K^0, for which $\text{dist}(x, y) = p$. Let us prove by induction that

$$\sum_d^d \leqslant \sum_1^d. \tag{9.3}$$

This is straightforward for $d = 2$, as the sum of the squares of the two diagonals of a quadrilateral is not more than the sum of the squares of the four sides. Next we apply the inequality $\Sigma_2^2 \leqslant \Sigma_1^2$ to $(1, d-1)$-subrectangulars (which have sides of length 1 and $d - 1$) in the d-dimensional cube. Then by summing over all $d2^{d-1}$ such rectangulars we get

$$\sum_1^d + \sum_{d-1}^d \leqslant d \sum_d^d. \tag{9.4}$$

Finally, we observe that the inequality $\Sigma_p^p \leqslant \Sigma_1^p$ for some integer p in the interval $1 < p < d$ yields

$$\sum_p^d \leqslant \frac{(d-1)!}{(p-1)!\,(d-p)!} \sum_1^d, \tag{9.5}$$

as there are exactly $d!2^{d-1}/p!(d-p)!$ pairs of points x and y in K^0 with $\text{dist}(x, y) = p$.

In particular, the inductive assumption $\Sigma_{d-1}^{d-1} \leqslant \Sigma_1^{d-1}$ gives $\Sigma_{d-1}^d \leqslant (d-1)\Sigma_1^d$, and using (9.4) we prove (9.3) for all d.

Observe that for any distance-decreasing map $F\colon K^0 \to \mathbf{R}^N$, inequality (9.3) implies the existence of a pair of points x and y in K^0 with $\text{dist}(x, y) = d$ such that $\text{dist}(F(x), F(y)) \leqslant \sqrt{d} = \text{dist}(x, y)/\sqrt{d}$.

Now if $\Sigma_{x\in K^0} f(x) = 0$, then by using the same argument as in the proof of the above Proposition (A), and summing up inequality (9.5) over $p = 1, \cdots, d$, we get

$$\sum_{x\in K^0} f^2(x) \leqslant \frac{1}{2}\sum_1^d, \tag{9.6}$$

which means that the combinatorial eigenvalue $\lambda_1 = \lambda_1(K^0)$ is $\geqslant 2$. In fact, one may have equality in (9.6) only for linear projections of $K^0 \subset \mathbf{R}^d$ on $\mathbf{R}$, and thus

$$\lambda_1(K^0) = 2 \quad \text{for all } d = 2, 3, \cdots.$$

The quantity Int d^2 for K^0 is

$$\sum_{K^0}\sum_{K^0}[\operatorname{dist}(x_1, x_2)]^2 = \sum_{p=0}^{d} p^2 \frac{d!2^d}{p!(d-p)!} = 2^{2d-2}d(d-1).$$

The role of dimension n now is played by the number d, as every vertex in K^1 has d adjacent edges. Thus we get for the second time (and, in fact, for the same reason) a lower bound for the distortion of the maps $K^0 \to \mathbf{R}^N$ for arbitrarily large N:

$$(\text{distor})^2 \geqslant (\lambda_1 \operatorname{Int} d^2)/(2d2^{2d}) = \frac{d+1}{4}.$$

The geometry of the finite space K^0 can be "transplanted" to a closed surface of genus $d2^{d-1} - 2^d + 1$ as follows. We assign a small 2-sphere to each point in K^0, and join some spheres by narrow tubes (handles) of unit length, which correspond to the edges of the 1-skeleton K^1. In this way we get a surface whose every map to $\mathbf{R}^N$ has distortion $\geqslant \sqrt{d}/2$, but it is difficult to control the eigenvalue λ_1 since the geometry becomes complicated when d edges of K^1 come to one vertex.

9.2. Iterated cubical graphs. We want to modify the complex K^1 in order to have only three edges at every vertex.

Start with an arbitrary graph (X, A), that is, a 1-dimensional simplicial complex with the set X of vertices and the set A of edges. For a function $f = f(x)$, $x \in X$, and a subset $X' \subset X$, we write

$$\|f\|^2(X') = \sum_{x\in X'} f(x)^2.$$

We denote by Df the function on A which assigns the difference $f(x_i) - f(x_j)$ to each edge $a = (x_i, x_j)$ between some vertices x_i and x_j. *The inverse first* *eigenvalue* of X, denoted by $\Lambda(X) = \lambda_1^{-1}(X)$, is the smallest number $\Lambda > 0$ for which the inequality

$$\|f\|^2(X) \leqslant \Lambda \|Df\|^2(A)$$

holds for all those functions $f\colon X \to \mathbf{R}$, whose average value is zero, i.e.,

$$q^{-1} \sum_{x \in X} f(x) = 0,$$

for $q = \#(X) =$ the number of vertices.

Observe that every function f on X with the average value $\bar{f}$ satisfies

$$\|f - \bar{f}\|^2(X) \leqslant \Lambda(X)\|Df\|^2(X), \tag{9.7}$$

$$\begin{aligned} \|f\|^2(X) &= \|f - \bar{f}\|^2(X) + \|\bar{f}\|^2(X) = \|f - \bar{f}\|^2(X) + q\bar{f}^2 \\ &\leqslant \Lambda(x)\|Df\|^2 + q\bar{f}^2. \end{aligned} \tag{9.8}$$

Next we assume that the graph (X, A) has *degree* d, that is, every vertex $x \in X$ has d adjacent edges, and we *compose* (X, A) with another graph (Y, B) for which $\#(Y) = d$ as follows.

First we take q isomorphic copies of the graph (Y, B) labelled by the vertices $x \in X$, call them $(Y(x), B(x))$, and take the disjoint union of these copies:

$$(Y^*, B^*) = (Y \times X, B \times X) = \bigcup_{x \in X} (Y(x), B(x)).$$

Then we attach to the graph (Y^*, B^*) some additional edges, called a^* for all $a \in A$; namely, for every edge $a \in A$ joining some vertices x and x' in X we choose some points $y = y(a) \in Y = Y(x)$ and $y' = y'(a) \in Y = Y(x')$, and join these pairs (y, y') by the edges a^*. We require every point $y \in Y(x)$ for all $x \in X$ to have exactly one edge a^* attached to it. If we assume the graph (Y, B) to admit a transitive group of automorphisms, then the resulting *composed* graph $(Y^*, B^* \cup A^*)$ is uniquely determined (up to an isomorphism).

Let us estimate the number $\Lambda(Y^*)$ in terms of $\Lambda(X)$ and $\Lambda(Y)$. Take an arbitrary function $F = F(y^*) = F(y, x)$ on $Y^* = Y \times X$ such that

$$\sum_{y^* \in Y^*} F(y^*) = 0,$$

and put

$$f(x) = \sum_{y \in Y(x)} F(y, x).$$

Take an edge $a \in A$ between some vertices x and x' in X, and write

$$f(x) - f(x') = P(a) + Q(a) + R(a),$$

where

$$P(a) = F(y(a), x) - F(y'(a), x'),$$
$$Q(a) = d^{-1} \sum_{y \in Y(x)} (F(y, x) - F(y(a), x)),$$
$$R(a) = d^{-1} \sum_{y \in Y(x')} F(y'(a), x') - F(y', x').$$

Then

$$(f(x) - f'(x))^2 \leqslant 2\left[P^2(a) + (Q(a) + R(a))^2\right],$$

$$\|Df\|^2(A) \leqslant 2\|Df\|^2(A^*) + 4d^{-1} \sum_{x \in X} \left[\sum\sum_{y, z \in Y(x)} (F(y, x) - F(z, x))^2 \right]$$
$$\leqslant 2\|DF\|^2(A^*) + 8 \sum_{x \in X} \Lambda(Y)\|DF\|^2(B(x))$$
$$\leqslant (2 + 8\Lambda(Y))\|DF\|^2(B^* \cup A^*).$$

Next, for every $x \in X$,

$$\|F\|^2(Y(x)) \leqslant \Lambda(Y)\|DF\|^2(B(x)) + df^2(x),$$

and then

$$\|F\|^2(Y^*) \leqslant \Lambda(Y)\|DF\|^2(B^*) + d\|f\|^2(X)$$
$$\leqslant \Lambda(Y)\|DF\|^2(B^*) + d\Lambda(X)\|Df\|^2(A)$$
$$\leqslant (\Lambda(Y) + 2d\Lambda(X) + 8\Lambda(X)\Lambda(Y))\|DF\|^2(B^* \cup A^*).$$

In the following we shall have $\Lambda(Y) \leqslant 2$. Then

$$\|F\|^2(Y^*) \leqslant (2 + \Lambda(X)(2d + 16)\|DF\|^2(B^* \cup A^*),$$

and so

(9.9) $$\Lambda(Y^*) \leqslant (2d + 16)\Lambda(X) + 2.$$

Now we apply the above consideration to the one-skeletons of the d-dimensional cubes, which are studied in the previous section; namely, we take $d_0 = 3$, $d_1 = 2^{3-1} = 4$, and, in general, $d_{i+1} = 2^{d_i - 1}$. Then we take for (X, A) the 1-skeleton of the d_k-dimensional cube, and for (Y, B) the 1-skeleton of $(d_{k-1} - 1)$-dimensional cube. Thus the composed graph $(Y^*, A^* \cup B^*)$ has degree d_{k-1}, and we take it for the new space X. The corresponding new Y is the 1-skeleton of the $(d_{k-2} - 1)$-dimensional cube, and this descending process is carried over until we arrive at a graph of degree three, which is called the *k-iterated cubical graph* (X_k, A_k).

As we know that the number Λ for cubes is $\frac{1}{2}$, the above inequality for the composed graphs leads to the following bound for $\Lambda(X_k)$.

The first (cubical) graph has certain degree $d = d_k$. the first composition process increases Λ by a factor $\leqslant 20d$. The new degree satisfies $d_{k-1} \leqslant \log d_k$, as long as $d_k \geqslant 1000$. Therefore the second composition process makes Λ less than

$$(20d)\,(20\log d)\tfrac{1}{2}.$$

we have Λ less than

$$(20d)\,(20\log d)(20\log\log d)\tfrac{1}{2},$$

and so on. In particular, as $k \to \infty$,

$$\Lambda(X_k)/d_k^{1+\varepsilon} \to 0,$$

for every fixed positive ε.

The number of the vertices in our graph is the product

$$\#(X_k) = 2^{d_k} d_k d_{k-1} \cdots d_1, \quad \text{for } d_1 = 4.$$

We need only the asymptotic relation

$$\#(X_k)/\left(2^{d_k} d_k^{-1+\varepsilon}\right) \to 0, \quad \text{for } k \to \infty,$$

which holds for every fixed $\varepsilon > 0$.

Finally, we measure the distance in X_k between pairs of points by the length of the shortest paths of edges. We estimate a lower bound of the quantity $\operatorname{Int} d^2 = \Sigma\Sigma \operatorname{dist}^2(x, y)$ as follows. For the 1-skeleton of the d-dimensional cube, $\operatorname{Int} d^2$ is $2^{2d-2}d(d-1)$. The first composition increases this number by a factor $\geqslant d^2$, as each vertex "divides" into d copies and the distances grow up. The further compositions may only increase $\operatorname{Int} d^2$ and so

$$\operatorname{Int} d^2(X_k) \geqslant 2^{2d-2}d^3(d-1).$$

Thus the relevant ratio

$$\lambda_1 \operatorname{Int} d^2/(\#X_k)^2$$

is greater than $d^{1-\varepsilon}$ for every positive ε and large k.

Now as the graph (X_k, A_k) (unlike the original cubical graph) has fixed degree $= 3$, we can control the geometry of surfaces obtained from the union of spheres $\bigcup_{x\in X_k} S_x^2$ by attaching handles "along" the edges $a \in A_k$.

We enumerate the relevant properties of such surfaces in the final

9.2.A. Proposition-Example. *There exists a sequence of closed surfaces V_d for some sequence of numbers $d = d_k \to \infty$ with the following list of properties*:

(1) *The surfaces V_d have locally bounded geometry*: *their sectional curvatures are pinched between* $+1$ *and* -1, *and their injectivity radii are everywhere greater*

than one. (*In fact, one even can make the sectional curvature arbitrarily close and probably equal to* -1.)

(2) *The genus* of V_d *is about* $2^d d$, *that* is, the *genus* is *pinched between* $2^d d^{1-\varepsilon}$ *and* $2^d d^{1+\varepsilon}$ *for any fixed* $\varepsilon > 0$ *and* $d \to \infty$.

(3) *The total area of* V_d *is about* $2^d d$ *in the same sense.*

(4) *The first eigenvalue of the Laplace operator on* V_d *is about* d^{-1}.

(5) *The integral*

$$\operatorname{Int} d^2 = \iint_{V \times V} \operatorname{dist}(v_1, v_2)\, dv_1\, dv_2$$

is about $2^{2d} d^4$.

(6) *The filling radii of the surfaces* V_d *are* $\leqslant$ const. *Moreover, each* V_d *bounds a handle-body* $B_d \supset V_d$ *such that the inclusion* $V_d \to B_d$ *is isometric* (*in particular, the boundary* $\partial B_d = V_d \subset B_d$ *is totally geodesic in* B_d), *and the distances* $\operatorname{dist}(b, V_d)$ *are uniformly bounded for all* $b \in B_d$ *and all* $d = d_k$. (*This handle-body is obtained by a three-dimensional thickening of the graph* (X_k, A_k).)

9.3. On the Ramsey-Dvoretzki-Milman phenomenon. The properties of the above "cubical" spaces illustrate the following general heuristic principle, called *Ramsey's phenomenon*: "If a function f on a *large* space X has small oscillation, then the function f has *very* small oscillation on *many* subspaces of X, and also has *very* small average oscillation relative to *many* measures on X."

Examples of Ramsey's phenomenon. We start with the classical

9.3.A. Ramsey's theorem. *Let* Δ *be an infinite dimensional simplex with countably many vertices, and let* $X = X^k(\Delta)$ *be the set of the barycenters of the* k*-dimensional faces of* Δ. *Let* f *be an arbitrary map of* X *into a finite set. Then there exists an infinite dimensional face* $\Delta' \subset \Delta$ *such that the map* f *is constant on the subset* $X' = X^k(\Delta') \subset X$. (See [30] for more combinatorial examples.)

9.3.A′. Milman's theorem. *Let* f *be a uniformly continuous function* (*for example, a Lipschitz function with dilation* $\leqslant \text{const} < \infty$) *on the unit sphere in the infinite dimensional Hilbert space* $f\colon S^\infty \to \mathbf{R}$. *Let* K *be an arbitrary compact subset in* S^∞, *and let* $\varepsilon > 0$ *be any given positive number. Then there exists an isometry* $\operatorname{Is}\colon K \to S^\infty$ *such that the composed function* $f \circ \operatorname{Is}\colon K \to \mathbf{R}$ *is* ε*-constant, that is,* $f \circ \operatorname{Is}$ *sends* K *to an* ε*-interval in* $\mathbf{R}$.

This is one of many generalizations of the famous theorem of *Dvoretzki* (see [29] for further information).

9.3.B. Wirtinger's inequality. *The unit sphere* S^n *has the first eigenvalue* $\lambda_1(S^n) = n$. *Therefore if a function* $f\colon S^n \to \mathbf{R}$ *has* $[\int_{S^n} \|\operatorname{grad} f(s)\|^2\, ds]/\operatorname{Vol} S^n = D$, *then*

$$\left[\int_{S^n} (f(s) - A)^2\, ds\right] \Big/ \operatorname{Vol} S^n \leqslant \frac{D}{n},$$

for

$$A = \left(\int_{S^n} f(s)\,ds\right)/\mathrm{Vol}\,S^n.$$

In particular, if Dil $f \leqslant 1$ *and* n *is large, then the function* f *is close to the average value* A *on a subset in* S^n *of almost full measure.*

There are close relations between the theorems of Milman and of Wirtinger. In fact, if a function f on S^n is *ε-constant* (i.e., Dil $f \leqslant \varepsilon$) on a subset $U \subset S^N$ of almost full measure, $\mathrm{mes}_n U \geqslant (1-\delta)\mathrm{Vol}\,S^n$, then by integral geometry there is a k-dimensional equator $S^k \subset S^n$ for any given $k \leqslant n$ such that

$$\mathrm{mes}_k(U \cap S^k) \geqslant (1-\delta)\mathrm{Vol}\,S^k. \tag{9.10}$$

If we further assume Dil $f \leqslant 1$, and let $n \to \infty$, then ε and δ converge to zero by Wirtingers' inequality. Then inequality (9.10) for any *fixed* k makes the function f almost constant on the whole equator S^k. As any uniformly continuous function can be approximated by Lipschitz functions, and every compact subset in S^∞ can be "approximated" by spheres $S^k \subset S^\infty$, we obtain the implication (9.3.B) ⇒ (9.3.A′).

This argument is due to Milman [60] who originally used more powerful *Levy's isoperimetric inequality* instead of Wirtinger's. (See [60], [53], [35], [40].)

Also observe that Milman's theorem generalizes the following simple classical fact: *An arbitrary n-dimensional linear space of functions on a probability space K contains a function* $f_0 \not\equiv 0$, *for which*

$$\|f_0\|_{L^2} \leqslant \|f_0\|_{L^\infty}/\sqrt{n}\,. \tag{9.1}$$

(Analogous relations hold among all norms $\|\ \|_{L^p}$.)

If K is a subset in $S^{n-1} \subset \mathbf{R}^n$, and the relevant space consists of *linear* functions, then (9.1) implies Milman's theorem for a normal linear projection $f\colon K \to \mathbf{R} \subset \mathbf{R}^n$, as there is some rotation Is of S^n such that the composition $f \circ \mathrm{Is}$ becomes L^2-close on K to a constant.

Let us give a simple proof of the following Riemannian version of Milman's theorem.

9.3.C. Theorem. *Let* V *be a closed Riemannian manifold of dimension* n, *let* X *be an arbitrary Riemannian manifold, and let* $f\colon V \to X$ *be a Lipschitz map which sends* V *onto an m-dimensional subset of* X *for some* $m \leqslant n$. *Then for any given number* $l > 0$ *there exists a geodesic segment* K_0 *in* V (*possibly with self intersections*) *of length* l *whose image has*

$$\mathrm{length}\, f(K_0) \leqslant l(\mathrm{Dil}\, f)\sqrt{m/n}\,.$$

Proof. The map f is almost everywhere differentiable, and the differential $D = D_v f\colon T_v(V) \to T_x(X)$, $x = f(v)$, has rank $\leqslant m$. Therefore the average of $\|D(s)\|^2$ over the unit sphere $S^{n-1} \subset T_V(V)$ satisfies

$$\left[\int_{S^{n-1}} \|D(s)\|^2\, ds\right] \Big/ \operatorname{Vol} S^{n-1} \leqslant \frac{m}{n} \sup_{s \in S^{n-1}} \|Ds\|^2 \leqslant \frac{m}{n} (\operatorname{Dil} f)^2.$$

It follows that the average of $\|D(s)\|^2$ over the Liouville measure on the unit tangent bundle $S(V)$ of V is also bounded by $\frac{m}{n}(\operatorname{Dil} f)^2$. As the Liouville measure is invariant under the geodesic flow, we get the same bound for the average $\operatorname{Av}_{\{K\}} I(K)$ of the integrals

$$I(K) = l^{-1} \int_K \|D(s_k)\|^2\, dk,$$

where s_k denotes the unit tangent vector to the geodesic segment K, and K runs over all segments of length l. Thus we have a segment K_0 for which $I(K_0) \leqslant \operatorname{Av}_{\{K\}} I(K) \leqslant \frac{m}{n}(\operatorname{Dil} f)^2$, and obviously

$$\operatorname{length} f(K_0) \leqslant l\sqrt{I(K_0)} \leqslant l \operatorname{Dil} f \sqrt{m/n}\,.$$

9.3.C′. Remark and corollaries. (a) If $V = S^n$ and $X = \mathbf{R}$, then we apply Milman's theorem for $K = S^1 \subset S^n \subset S^\infty$ by letting $n \to \infty$. In fact, the general case of Milman's theorem can also be derived along these lines (see below).

(b) The above proof also applies to those *noncompact* manifolds V which admit on averaging operator on bounded functions on V. Such manifolds are, for instance, complete manifolds V of *subexponential growth* which means that balls around a fixed point $v \in V$ have

$$\lim_{R \to \infty} R^{-1}(\log \operatorname{Vol} B(R)) \to 0.$$

For example, the above theorem holds for $V = \mathbf{R}^n$. However, the theorem is false for the hyperbolic space H^n for $n \geqslant 10$, as the distance function $f = f(v) = \operatorname{dist}(v, v_0)$ to a fixed point has oscillation $\geqslant l/3$ on every geodesic segment in H^n of length $l \geqslant 100$.

Now let V be a complete Riemannian manifold of arbitrary dimension ($\geqslant n$), and let a locally compact group G act isometrically on V. Let G_v, $v \in V$, denote the isotropy subgroup of a point v, and let the (linear) action of G_v on the tangent space $T_v(V)$ has no nontrivial invariant subspaces of dimension $\leqslant n$ for all points $v \in V$. Let $f\colon V \to X$ be a Lispchitz map of V onto an m-dimensional subset in a Riemannian manifold X.

9.3.C″. Theorem. *Let $K \subset V$ be a piecewise smooth one-dimensional subcomplex in V of total length l such that every two points in K can be joined by a curve of length $\leqslant l_0 \leqslant l$ in K. Then in the following two cases there exists an isometry $g \in G$ of V such that*

$$\text{length}(f \circ g)(K) \leqslant l(\text{Dil}\, f)\sqrt{m/n}\,,$$

$$\text{Osc}\, f \mid g(K) \leqslant (\text{Dil}\, f)\sqrt{l_0 l_m/n}\,.$$

(1) *The group G is amenable. For example, G is an extension of a compact group by a solvable group.*

(2) *The function f is invariant under a discrete subgroup $\Gamma \subset G$ for which the quotient V/Γ has subexponential growth.*

Proof. As the invariant subspaces of the action G_v on T_v have dimensions $\geqslant n$, by linear algebra one gets the following bound for the average of $\| D(gs)\|^2$ over the group G_v with the normalized Haar measure, where D is the differential of f at $v \in V$, and s is a vector in the unit sphere $S_v \subset T_v(V)$:

$$\int_{G_v} \|D(gs)\|^2\, dg \leqslant \frac{m}{n} \sup_{s \in S_v} \|D(s)\|^2.$$

Conditions (1) and (2) allow one to average the above inequality over the group G, so that one gets a translate $K_0 = gK$ for some $g \in G$, for which the restriction $f \mid K_0$ has

$$I(K_0) = l^{-1} \int_{K_0} \|D(s_k)\|^2\, dk \leqslant \frac{m}{n}(\text{Dil}\, f)^2,$$

(compare Theorem 9.3.C). Thus

$$\text{length}\, f(K_0) \leqslant l\sqrt{I(K_0)} \leqslant l\,\text{Dil}\, f\sqrt{m/n}\,,$$

as well as

$$\text{Osc}\, f \mid K_0 \leqslant \sqrt{l_0 l I(K_0)} \leqslant \text{Dif}\, f\sqrt{l_0 l_m/n}\,.$$

Corollary. *Let V, X and f satisfy the assumption of Theorem 9.3.C″, and let $K' \subset V$ be a finite subset which contains $q + 1$ points: $K' = \{v_0, \cdots, v_q\}$. Then for some translate $K_0' = gK'$*

$$\text{Osc}\, f \mid K_0' \leqslant \text{Dil}\, f(\text{Diam}\, K')\sqrt{qm/n}\,.$$

Proof. Apply the theorem to the union of minimal geodesic segments between v_0 and v_i, $i = 1, \cdots, q$.

Theorem 9.3.C″ and the above corollary are most interesting when applied to sequences of G-manifolds $(V, G)_n$ for $n \to \infty$. However, some infinite dimensional ($n \to \infty$) results can be obtained directly as the following analysis shows.

Let L be a Banach space, and let L_1 and L_2 be linear subspaces in L. We say that L_1 and L_2 are *δ-orthogonal* if the restrictions to L_1 and L_2 of an arbitrary linear functional $a: L \to \mathbf{R}$ satisfy

$$\frac{\|a \mid L_1\| + \|a \mid L_2\|}{2} \leqslant (1-\delta)\|a\|.$$

For example, orthogonal subspaces in a Hilbert space L are δ-orthogonal for all $\delta \leqslant 1 - 1/\sqrt{2}$.

A family of linear operators $\{A\}$, $A: L \to L$, is called *δ-nonrecurrent*, if for every finite dimensional subspace L_1 in L there exists an operator $A \in \{A\}$ such that the image $L_2 = AL_1$ is δ-orthogonal to L_1.

Examples. (a) Let A be a unitary (i.e., isometric) operator on a Hilbert space L. If A has continuous spectrum (i.e., there is no invariant finite dimensional subspaces), then the powers $\{A^d\}$, $d = 1, \cdots$, of A form a δ-nonrecurrent family for every $\delta < 1 - 1/\sqrt{2}$.

(b) Let L be the l^p-space of functions $f: \mathbf{Z} \to \mathbf{R}$, $\|f\| = (\Sigma_i |f(i)|^p)^{1/p}$, and let A be the shift operator

$$A: f(i) \mapsto f(i+1).$$

Then the family $\{A^d\}$, $d = 1, \cdots$, is δ-nonrecurrent for every $\delta < 1 - 2^{1-1/p}$.

A family of operators is said to be *nonrecurrent* if it is δ-nonrecurrent for some *positive* $\delta > 0$.

We assign to a family of operators $\{A\}$ on L the following set $G = G\{A\}$ of affine maps g of L into itself: $G = \{g: x \to Ax + y\}$ for all $A \in \{A\}$ and all $y \in L$.

9.3.D. Theorem. *Let $\{A\}$ be a nonrecurrent family of uniformly bounded operators (i.e., $\|A\| \leqslant \text{const} < \infty$ for all $A \in \{A\}$) on a Banach space L. Let f: $L \to \mathbf{R}$ be a uniformly continuous function, and let $K \subset L$ be an arbitrary compact subset. Then for every $\varepsilon > 0$ there exists a transformation g: $L \to L$ in $G\{A\}$ such that the function f is ε-constant on the image $K_0 = g(K)$, i.e.,*

$$\operatorname{Osc} f \mid K_0 \leqslant \varepsilon.$$

Proof. We assume without loss of generality that the set K is finite, $K = \{v_0, \cdots, v_q\}$, and we put $x_i = (v_i - v_0)/\|v_i - v_0\|$, $i = 1, \cdots, q$.

As the family $\{A\}$ is nonrecurrent and uniformly bounded, for every $\varepsilon' > 0$ there exist some operators $A_j \subset \{A\}$, $j = 1, \cdots, n$, such that every linear functional $a: L \to \mathbf{R}$ satisfies

$$\frac{1}{n} \sum_{j=1}^{n} a\big(A_j(x_i)\big) \leqslant \varepsilon' \|a\|, \tag{9.12}$$

for all $i = 1, \cdots, q$.

Let $L' = L'(\varepsilon')$ be the span of the vectors $A_j(x_i)$, $j = 1,\cdots,n$, $i = 1,\cdots,q$. The restriction of the function f to the *finite dimensional* space L' can be approximated by Lipschitz functions whose dilations can be controlled by the chosen precision of the approximation and the modulus of continuity of f. Thus we may assume the function f to be Lipschitz on L' such that the dilation dil fL' is independent of ε'.

Let $g_{j,v}$: $x \to A_j x + v$, for $j = 1,\cdots,n$ and $v \in L'$. Denote by $\bar{K}$ the union of the straight segments $[v_0, v_i]$ over $i = 1,\cdots,q$, and let $\bar{K}(j, v) = g_{j,v}(\bar{K}) \subset L'$. Then we restrict the function f to the set $\bar{K}(j, v)$, and denote by $\bar{I}(j, v)$ the integral of $\| D(f | \bar{K}(j, v) \|$ over $\bar{K}(j, v)$. According to (9.12) the average value of $\bar{I}(j, v)$ over $j = 1,\cdots,n$ and over $v \in L'$ is bounded by ε'(Dil f) length $\bar{K}$. Therefore there is some set $\bar{K}(j_0, v_0)$ for which $\bar{I}(j_0, v_0) \leqslant \varepsilon'$(Dil f) length $\bar{K}$. By choosing $\varepsilon' \leqslant \varepsilon/$(Dil $| f$) length $\bar{K}$, we obtain

$$\text{dil}\, f |\, g_{j_0,v_0}(K) \leqslant I(j_0, v_0) \leqslant \varepsilon.$$

Appendix 1. Slicing and mapping invariants of Riemannian manifolds

Let us define *the diameter* of a map f: $X \to Y$ between two metric spaces as follows:

$$\text{Diam}\, f = \sup_{x_1, x_2 \in X} \left[\text{dist}(x_1, x_2) - \text{dist}(f(x_1), f(x_2))\right].$$

For example, if f is constant, then Diam f = Diam X. If f: $X \to X$ is the identity map, then Diam $f = 0$.

(A) Lemma. *If* Diam $f = \delta < \infty$, *then there exists a continuous map* Q: $Y \to L^\infty(X)$, *such that the distance between the composed map* $Q \circ f$: $X \to L^\infty(X)$ *and the canonical imbedding* I: $X \to L^\infty(X)$ *equals* $\delta/2$, *i.e.*,

$$\text{dist}(Q \circ f(x), I(x)) = \delta/2, \quad \textit{for all } x \in X.$$

Proof. Send every point $y \in Y$ to the following function $Q_y(x) \in L^\infty(X)$:

$$Q_y(x) = \delta/2 + \inf_{x' \in X} \left[\text{dist}(x, x') + \text{dist}(y, f(x'))\right].$$

Recall that a continuous map f: $X \to Y$ is said to be *k-degenerate* if it factors: $f = f' \circ f''$, through a k-dimensional polyhedron k for some continuous maps f': $K \to Y$ and f'': $X \to K$. Then we introduce the *k-diameter* of X:

$$\text{Diam}_k\, X = \inf_{f, Y} \text{Diam}\, f,$$

over all metric spaces Y and all k-degenerate maps f: $X \to Y$.

Example. If V is a connected n-dimensional polyhedron, then

$$\text{Diam}\, V = \text{Diam}_0\, V \geqslant \text{Diam}_1\, V \geqslant \cdots \geqslant \text{Diam}_{n-1}\, V \geqslant \text{Diam}_n\, V = 0.$$

There is an alternative definition of these diameters Diam_k, with another notion of a diameter for maps $f\colon X \to Y$; namely, take the pullbacks $f^{-1}(y) \subset X$ and put

$$\mathrm{Diam}' f = \sup_{y \in Y} \mathrm{Diam}\, f^{-1}(y).$$

Then define

$$\mathrm{Diam}'_k X = \inf_{f,K} \mathrm{Diam}' f,$$

over all k-dimensional polyhedra K and all continuous maps $f\colon X \to K$. It is clear that

$$\mathrm{Diam}' f \leqslant \mathrm{Diam}\, f,\ \mathrm{Diam}'_k X \leqslant \mathrm{Diam}_k X,$$

and that every locally compact n-dimensional metric space X has

$$\mathrm{Diam}'_k X > 0, \quad \text{for } k < n,$$
$$\mathrm{Diam}'_k X = 0, \quad \text{for } k \geqslant n.$$

(B) Lemma. *Let V be an n-dimensional polyhedron with a complete piecewise Riemannian metric. Then*

$$\mathrm{Diam}'_k V = \mathrm{Diam}_k V, \quad \textit{for all } k = 0, \ldots.$$

Proof. Indeed, if $f\colon V \to K$ is a continuous map, for which $\mathrm{Diam}' f < \delta < \infty$, then there exists a (sufficiently large) metric in K, relative to which

$$\mathrm{Diam}\, f < \delta.$$

Next we introduce the k-radius $\mathrm{Rad}_k(V \subset X)$ of an embedding $I\colon V \subset X$ as a lower bound of those $\varepsilon \geqslant 0$, for which there exists a *k-degenerate* map $f\colon V \to X$ within distance ε from I, i.e.,

$$\mathrm{dist}(f(v), I(v)) \leqslant \varepsilon \quad \text{for all } v \in V,$$

(compare Corollary 3.1.A′ and Examples 3.1.A″). We define, in particular,

$$\mathrm{Rad}_k V = \mathrm{Rad}_k(V \subset L^\infty(V)),$$

for the canonical embedding $V \subset L^\infty(V)$.

Example. Let V be a complete Riemannian manifold. Then

$$\mathrm{Rad}_k V \geqslant \mathrm{Fill\,Rad}\, V, \quad \text{for all } k < \dim V.$$

(C) Question. Does the following inequality hold with some universal constant const $=$ const(n), for $n = \dim V$?

$$\mathrm{Rad}_{n-1} V \leqslant \mathrm{const}(\mathrm{Vol}\, V)^{1/n}.$$

(D) Proposition. *Let V be a complete Riemannian manifold or a piecewise Riemannian polyhedron. Then*

$$\mathrm{Rad}_k V = \tfrac{1}{2}\mathrm{Diam}_k V = \tfrac{1}{2}\mathrm{Diam}'_k V,$$

for all $k = 0, \cdots$.

Proof. The inequality $\mathrm{Rad}_k \geqslant \frac{1}{2}\mathrm{Diam}_k$ follows from the obvious inequality

$$\mathrm{Diam}\, f \leqslant 2\mathrm{dist}(f, I),$$

for the canonical embedding I: $V \subset L^\infty(V)$ and an arbitrary map f: $V \to V^\infty(V)$. The inequality $\mathrm{Rad}_k \leqslant \frac{1}{2}\mathrm{Diam}_k$ follows from Lemma (A).

(E_1) Examples. *Let V be a connected surface of genus zero with an arbitrary complete Riemannian metric. Then*

$$\mathrm{Rad}_1 V = \tfrac{1}{2}\mathrm{Diam}'_1 V \leqslant (\mathrm{Area}\, V)^{1/2}.$$

(Thus we obtain the positive answer to the above question (C) for surfaces V of genus zero.)

Proof. We start with the following factorization of an arbitrary proper map f: $X \to Y$ between locally compact metric spaces. First we partition the space X into *connected components* of the pullbacks $f^{-1}(y) \subset X$ for all $y \in Y$. Denote

$$\tilde{Y} = X/\,(\text{the partition}),$$

and consider the quotient map $\bar{f}$: $X \to \tilde{Y}$. The new pullbacks $\bar{f}^{-1}(\tilde{y}) \subset X$ are exactly the connected components of the pullbacks $f^{-1}(y)$. Then there is a unique map $\tilde{f}$: $\tilde{Y} \to Y$ such that $f = \tilde{f} \circ \bar{f}$.

(E'_1) Definition. The map $\bar{f}$: $X \to \tilde{Y}$ is called the *connected* map associated to f.

(E''_1) Lemma. *Let X be a connected n-dimensional manifold such that the intersection index between every two homology classes of dimensions one and $n-1$ is zero, i.e., $H_1(X) \cap H_{n-1}(X) = 0$. Then, for an arbitrary Morse function f: $X \to \mathbf{R}$, the corresponding space $\tilde{Y} = \tilde{\mathbf{R}}$ is contractible. As this space $\tilde{Y}$ is one-dimensional, it is called the tree of the function f.*

Proof. As the map $\bar{f}$: $X \to \tilde{Y}$ is connected, every simple closed curve (a cycle) in the graph $\tilde{Y}$ lifts to a closed curve in X. Such a lift necessarily has a nonzero intersection with a pullback $\bar{f}^{-1}(\tilde{y}) \subset X$ for some $\tilde{y} \in \tilde{Y}$.

Now we consider the distance function $f(v) = \mathrm{dist}(v, v_0)$ to a fixed point v_0 in our surface V. Take the associated connected map $\bar{f}$: $V \to \tilde{Y} = \tilde{\mathbf{R}}$, and let us show that

$$\delta(\tilde{y}) = \mathrm{Diam}\, \bar{f}^{-1}(\tilde{y}) \leqslant 2(\mathrm{Area}\, V)^{1/2}, \quad \text{for all } \tilde{y} \in Y.$$

Since V has genus zero, the space $\tilde{Y}$ is a tree. (Strictly speaking, Lemma (E''_1) does not apply, as f is not a Morse function. However, one can approximate f

by a Morse function, and then (E_1'') does apply.) Therefore for every positive number $\rho \leqslant \frac{1}{2}\delta(\tilde{y})$ there exists a point $\tilde{y}_\rho \in \tilde{Y}$ such that $\tilde{f}(\tilde{y}) - \tilde{f}(\tilde{y}_\rho) = \rho$, and the pullback $C = \tilde{f}^{-1}(\tilde{y})$ in V is contained in the closed ρ-neighborhood of $C_\rho = \tilde{f}^{-1}(\tilde{y}_\rho)$. Thus, the curve C which is a component of the level $f^{-1}(t)$ for $t = \tilde{f}(\tilde{y})$), is preceded by the curves C_ρ, for which $f \mid C_\rho \equiv t - \rho$ and $\operatorname{Diam} C_\rho \geqslant \delta(\tilde{y}) - 2\rho$. As length $C_\rho \geqslant 2 \operatorname{Diam} C_\rho \geqslant \delta(\tilde{y}) - 2\rho$, according to the coarea formula the union of these curves, $V_\delta = \bigcup_{0 \leqslant \rho \leqslant \delta/2} C_\rho$, for $\delta = \delta(\tilde{y}) = \operatorname{Diam} C$ satisfies

$$\operatorname{Area} V \geqslant \operatorname{Area} V_\delta \geqslant \int_0^{\delta/2} 2\rho \, d\rho = \delta^2/4.$$

Thus we obtained the required estimate for the 1-*degenerate map* $\tilde{f}$: $V \to \tilde{Y}$:

$$\operatorname{Diam}' \tilde{f} \leqslant 2(\operatorname{Area} V)^{1/2}.$$

Remark. The above argument also applies to complete surfaces V of positive genus $g < \infty$, but the conclusion is weaker:

$$\operatorname{Rad}_1 V = \tfrac{1}{2}\operatorname{Diam}_1' V \leqslant (g+1)(\operatorname{Area} V)^{1/2}.$$

(E_2) *Let V be a complete Riemannian manifold such that every simple closed curve $C \subset V$ has*

$$\operatorname{Fill\,Rad}(C \subset V) \leqslant \rho_0 < \infty.$$

Then

$$\operatorname{Rad}_1 V = \tfrac{1}{2}\operatorname{Diam}_1' V \leqslant 3\rho_0$$

Proof. It suffices to show that every connected component $\tilde{V}_t$ of the level $f^{-1}(t)$ of the function $f(v) = \operatorname{dist}(v, v_0)$ has $\operatorname{Diam} \tilde{V}_t \leqslant 6\rho_0$ for all $t \in [0, \infty]$. To see this, we join a pair of points v_1 and v_2 in $\tilde{V}_t$ by a curve $\tilde{\gamma}$ in $\tilde{V}_t$, and let γ_1 and γ_2 be minimal segments between v_0 and the points v_1 and v_2 respectively. Then the closed curve $C = \gamma_1 \circ \gamma \circ \gamma_2^{-1}$ has $\operatorname{Fill\,Rad} C \geqslant \frac{1}{6}\operatorname{dist}(v_1, v_2)$. Indeed any filling S of C contains a point v for which

$$\operatorname{dist}(v, \gamma) = \operatorname{dist}(v, \gamma_1) = \operatorname{dist}(v, \gamma_2) = \rho.$$

Thus by the triangle inequality we have $\rho \geqslant \frac{1}{6}\operatorname{dist}(v_1, v_2)$, and as $\rho \geqslant \operatorname{Fill\,Rad} C$ the proof is concluded.

(E_2') Corollary. *Let V be a complete 3-dimensional manifold of positive scalar curvature so that*

$$\operatorname{Scal}(V) \geqslant \sigma^2 > 0.$$

If the intersection index on the homology of V vanishes, i.e., if $H_1(V) \cap H_2(V) = 0$ (compare Lemma E_1''), then

$$\operatorname{Rad}_1 V = \tfrac{1}{2}\operatorname{Diam}_1' V \leqslant \frac{\pi 3\sqrt{2}}{\sigma}.$$

Proof. Indeed,

$$\text{Fill Rad}(C \subset V) \leqslant \frac{\pi\sqrt{2}}{\sigma},$$

for all closed curves C in the above manifold V (see [39]).

Question. Let V be a complete n-dimensional manifold of positive scalar curvature $\geqslant \sigma^2 > 0$. Is it true that

$$\text{Rad}_{n-2} V \leqslant \text{const}_n/\sigma,$$

or at least that

$$\text{Fill Rad } V \leqslant \text{const}_n/\sigma?$$

Using the present state of knowledge even one can not exclude the possibility of the manifold V being geometrically contractible (see §4.5.D).

(E_3) *Let V be a complete geometrically contractible manifold of dimension n; for instance, $V = \mathbf{R}^n$. Then*

$$\text{Rad}_{n-1} \geqslant \text{Fill Rad } V = \infty,$$

(see Theorem 4.5.D′). Therefore for every continuous map f: $V \to \mathbf{R}^{n-1}$, there exists a *connected component* $\tilde{V}_y$ of the pullback $f^{-1}(y) \subset V$ of some point $y \in \mathbf{R}^{n-1}$, which has arbitrarily large diameter. This is seen by passing to the associated connected map $\tilde{f}$: $V \to \tilde{Y} = \tilde{\mathbf{R}}^{n-1}$.

(E_4) *Let V be a complete manifold whose Ricci curvature is bounded from below by* -1:

$$\text{Ricci } V \geqslant -1.$$

Then there exists a positive number $\varepsilon = \varepsilon(n) > 0$ *for* $n = \dim V$ *such that the inequality*

$$\text{Vol } B_v(1) \leqslant \delta^n \leqslant \varepsilon^n$$

for all unit balls $B_v(1)$ in V implies the inequality

$$\text{Rad}_{n-1} V \leqslant \text{const}_n\, \delta,$$

for some universal constant $\text{const}_n \geqslant 0$.

In fact, there is a map f of V to some $(n-1)$-dimensional subcomplex of the nerve of certain covering of V by small balls such that

$$\text{Diam}' f \leqslant \text{const}_n\, \delta,$$

(see [32]).

(E_4') Corollary. *Let the sectional curvature of V be bounded, i.e.,* $|\text{Curv } V| \leqslant 1$. *If the injectivity radius of V is "small" everywhere, i.e., if* $\text{Inj Rad } V \leqslant \delta \leqslant \varepsilon(n)$, *then*

$$\text{Rad}_{n-1} V \leqslant \text{const}_n'\, \delta.$$

In fact, as shown in [37], the small injective radius condition implies that V is close, in the Hausdorff metric, to a certain $(n-1)$-dimensional space. This leads (see Appendix 3) to an alternative proof of the implication

$$(\text{Inj Rad} \to 0) \Rightarrow (\text{Rad}_{k-1} \to 0).$$

Question. Is the assumption Ricci $V \geqslant -1$ essential for the conclusion of the proposition (E_4)?

(E_5) *Let V be a complete connected Riemannian manifold, and let $\varphi\colon V \to V$ be an arbitrary continuous map. If the intersection $H_1(V) \cap H_{n-1}(V)$ is zero for $n = \dim V$ (compare (E_1'')), then in the following three cases there exists a point $v_0 \in V_0$ for which*

$$\text{dist}(v_0, \varphi(v_0)) \leqslant 2\,\text{Rad}_1 V = \text{Diam}_1' V.$$

(1) *V is compact.*

(2) *V is connected at infinity, and the map f is onto.*

(3) *The map f is proper, and the canonical (continuous) extension of φ to the space of ends of V, $\varphi_*\colon \text{End}_V \to \text{End}_V$, has no fixed points.*

Proof. According to Lemma (E_1'') we have a continuous *connected* map f of V onto a *tree* Y such that $\text{Diam}' f \leqslant \delta$ where one may take δ as close to $\text{Diam}' V$ as one wishes. Then one has the following *set-valued* map $\Phi = f \circ f^{-1}\colon Y \to Y$. This map is obviously closed (i.e., its graph in $Y \times Y$ is closed), and every set $\Phi(y) \subset Y$ for all $y \in Y$ is contractible as the map f is connected. Therefore under the assumptions (1)–(3) there exists a "fixed" point $y_0 \in Y$ for which $\Phi(y_0) \ni y_0$. This means that the image $f(V_0) \subset V$ for the level $V_0 = f^{-1}(y_0)$ intersects V_0. Then some point $v_0 \in V$ has $f(v_0)$ in V_0, and so

$$\text{dist}(v_0, f(v_0)) \leqslant \text{Diam}\, V_0 \leqslant \delta.$$

(E_5') Corollaries. (a) *Let V be homeomorphic to S^2. Then every continuous map $\varphi\colon V \to V$ admits a point $v_0 \in V$ for which*

$$\text{dist}(v_0, \varphi(v_0)) \leqslant 2(\text{Area}\, V)^{1/2}.$$

(This generalizes a result by Berger; see [13].)

(a′) *The conclusion of (a) holds if V is homeomorphic to $\mathbf{R}^2$ and the map φ is onto.*

(b) *Let V be homeomorphic to S^3, and let the scalar curvature of V be positive and $\geqslant \sigma^2$. Then every continuous map $\varphi\colon V \to V$ admits a point v_0 for which*

$$\text{dist}(v_0, \varphi(v_0)) \leqslant \pi 6\sqrt{2}/\sigma.$$

(b′) *The conclusion of (b) holds if V is homeomorphic to $S^2 \times \mathbf{R}^1$, the map φ is proper and interchanges the two ends of V.*

(F) Volumes of maps. Let V be an n-dimensional Riemannian manifold, and let $f\colon V \to P$ be a continuous map into a $(n-m)$-dimensional space P. Let the pullbacks (slices) $f^{-1}(p) \subset P$ have finite m-dimensional Hausdorff measure, i.e.,

$$\operatorname{Vol}_m f^{-1}(p) < \infty,$$

for every point $p \in P$. Put

$$\operatorname{Vol}_m f = \sup_{p \in P} \operatorname{Vol}_m f^{-1}(p).$$

We want to express a lower bound of this volume $\operatorname{Vol}_m f$ in terms of geometric invariants of the manifold V.

(F_1) Let P be the real line, i.e., let $P = \mathbf{R}$. Suppose that the manifold V is compact, and consider *the Levi mean* of f, that is, the value $p_0 \in \mathbf{R}$ for which $\operatorname{Vol} f^{-1}(-\infty, p_0) = \frac{1}{2}\operatorname{Vol} V$. If V is isometric to the unit sphere S^n, then the classical isoperimetric inequality implies that

$$\operatorname{Vol}_{n-1} f \geqslant \operatorname{Vol}_{n-1} f^{-1}(p_0) \geqslant \operatorname{Vol} S^{n-1}.$$

Furthermore, if V is the unit ball in $\mathbf{R}^n$, then

$$\operatorname{Vol}_{n-1} f^{-1}(p_0) \geqslant \operatorname{Vol}(\text{unit ball in } \mathbf{R}^{n-1}).$$

(F_1') *If the manifold V is closed, and* Ricci $V \geqslant -(n-1)$, *then*

$$\operatorname{Vol}_{n-1} f \geqslant \operatorname{Vol}_{n-1} f^{-1}(p_0) \geqslant (\operatorname{Vol} V) \Big/ \left(2\int_0^D (\cos ht)^{n-1}\, dt\right),$$

for $D = \operatorname{Diam} V$ (*see* [35]).

(F_1'') Suppose that V is an orientable closed manifold, and let the function $f\colon V \to \mathbf{R}$ be smooth. Then the pullbacks $f^{-1}(p) \subset V$ of the regular values p of f form a family of submanifolds, and are continuous in p relative to the *flat norm* in the space of integral cycles. Indeed, any two cycles $f^{-1}(p_1)$ and $f^{-1}(p_2)$ in V bounds the chain $f^{-1}[p_1, p_2]$ whose volume becomes arbitrary small for $|p_1 - p_2| \to 0$. Then by the Almgren-Morse theory (compare Fact 1 of §8.1) there exists an $(n-1)$-dimensional minimal subvariety V_0 in V, for which

$$\operatorname{Vol} V_0 \leqslant \operatorname{Vol}_{n-1} f.$$

Now the volume of a minimal subvariety V_0 in V can be bounded from below by various invariants of V. For example, if Ricci $V \geqslant -(n-1)$, then (compare (F_1'))

$$\operatorname{Vol} V_0 \geqslant (\operatorname{Vol} V) \Big/ \left(\int_0^D (\cos ht)^{n-1}\, dt\right).$$

Another estimate is possible for manifolds V, which have sectional curvature $\leqslant \kappa^2$ and the injectivity radius $\geqslant \pi/2\kappa$, namely,

$$\operatorname{Vol} V_0 \geqslant \tfrac{1}{2}\operatorname{Vol} S^{n-1}/\kappa^{n-1},$$

for the unit sphere S^{n-1}.

(F_1''') *Let V be complete, and let f: $V \to \mathbf{R}$ be a smooth proper function. Then*

$$\operatorname{Vol}_{n-1} f \geqslant \operatorname{const}_n^*(\operatorname{Fill} \operatorname{Rad} V)^{n-1},$$

for some universal constant $\operatorname{const}_n^* > 0$.

Proof. Take a discrete set of regular values $\cdots, p_{-1}, p_0, \cdots, p_i, \cdots$ in $\mathbf{R}$ such that the regions $f^{-1}[p_i, p_{i+1}]$ have a (uniformly) small volume. Then we fill in each manifold $V_i = f^{-1}(p_i) \subset V$ (see §4.3) by a chain c_i, which is ε-close to V_i for $\varepsilon \approx \operatorname{Fill} \operatorname{Rad} V_i \leqslant \operatorname{const}_{n-1}(\operatorname{Vol}_{n-1} f)^{n-1}$, and whose n-dimensional volume is small so that

$$\operatorname{Vol} c_i \approx \operatorname{Fill} \operatorname{Vol} V_i \leqslant C_{n-1}(\operatorname{Vol}_{n-1} f)^{n/(n-1)}.$$

Thus we decompose V into the following sum of n-dimensional cycles: $V = \Sigma_{-\infty}^{\infty} z_i$ for

$$z_i = c_i + f^{-1}[p_i, p_{i+1}] - c_{i+1}.$$

Each cycle z_i has

$$\operatorname{Vol} z_i \lesssim \operatorname{Vol} c_i + \operatorname{Vol} c_{i+1} \leqslant 2C_{n-1}(\operatorname{Vol}_{n-1} f)^{n/(n-1)},$$

and is filled by a chain c_i' within distance ε' from z_i for $\varepsilon' = \operatorname{Fill} \operatorname{Rad} z_i \leqslant \operatorname{const}_n(\operatorname{Vol} z_i)^{1/n}$. So we get a filling of V within distance $\varepsilon + \varepsilon'$ from V.

(F_2) Let us generalize (F_1'') and (F_2''') to maps f of V into an $(m-n)$-dimensional manifold P.

(F_2') Suppose that the manifold V is closed and orientable, and let the map f: $V \to P$ be smooth. We look again at the family of cycles $f^{-1}(p)$ in V for the regular values p in P. In order to apply the Almgren-Morse theory we need the continuity (in p) of this family relative to the flat norm. It is likely that the condition $\operatorname{Vol}_m f < \infty$ alone implies this continuity. In any case, the continuity is obvious for all "decent" maps f. For example, the family $\{f^{-1}(p)\}$ is continuous if f is a *generic* C^∞-map or a real analytic map. To show this we must join any pair of cycles $f^{-1}(p_1)$ and $f^{-1}(p_2)$ by an $(m+1)$-dimensional chain $c = c(p_1, p_2)$ in V such that $\operatorname{Vol}_{m+1}(c) \to 0$ as $\operatorname{dist}(p_1, p_2) \to 0$. One gets such a chain by joining the points p_1 and p_2 by a generic short line segment γ with $c = f^{-1}(\gamma)$.

Now the Almgren-Morse theory yields an m-dimensional *minimal* subvariety V_0 in V, which has $\operatorname{Vol} V_0 \leqslant \operatorname{Vol}_m f$. The volume of V_0 has a lower bound:

$$\operatorname{Vol} V_0 \geqslant \operatorname{Vol} S^m/2\kappa^m,$$

provided V has sectional curvature $\leqslant \kappa^2$ and the injectivity radius $\geqslant \pi/2\kappa$. So we have

$$\mathrm{Vol}_m f \geqslant \tfrac{1}{2}\mathrm{Vol}\, S^m/\kappa^m,$$

for real analytic and generic C^∞-maps f: $V \to P$.

(F_2'') Let f: $V \to P$ be a smooth proper map. Suppose that for every $\delta > 0$ there exists a smooth triangulation of P, whose simplices Δ are transversal to f and their pullbacks have

$$\mathrm{Vol}_{m+k} f^{-1}(\Delta) \leqslant \delta,$$

for every k-dimensional simplex Δ and all $k = 1,\cdots,n - m$. This property is obviously satisfied by real analytic maps and generic C^∞-maps f.

Now we take the pullbacks $f^{-1}(p_i)$ of all vertices of our triangulation, and fill them in by some $(m + 1)$-dimensional chains c_i as in (F_1'''). If two vertices, say p_i and p_j in P, are joined by an edge $\Delta^1 = \Delta^1_{ij}$ in P, then we consider the $(m + 1)$-dimensional cycle

$$z_{ij} = c_i + f^{-1}(\Delta^1) - c_j,$$

whose volume is roughly equal (δ is small!) to

$$\mathrm{Vol}(c_i - c_j) \leqslant \text{Fill Vol}\, f^{-1}(p_i) + \text{Fill Vol}\, f^{-1}(p_j) \leqslant C_m(\mathrm{Vol}_m f)^{(m+1)/m}.$$

Then we fill each cycle z_{ij} by a "small" (compare (F_1''')) chain c_{ij}. Next we consider 2-symplices $\Delta^2 = \Delta^2_{ijk}$ in P. The sum of the pullback $f^{-1}(\Delta^2)$ with the chains c_{ij}, c_{ik} and c_{jk} is a cycle, say z_{ijk}, which then is filled in by a "small" chain c_{ijk}. We continue this process up to dimension $n - m = \dim P$, and thus we get a filling of the manifold V within a controlled distance from V. *Therefore*

$$\text{(A.1)} \qquad \text{Fill Rad}\, V \leqslant \mathrm{const}_n^*(\mathrm{Vol}_m f)^{1/m}$$

for above maps f: $V \to P$.

Remarks. (a) Most of the conditions which we have imposed on the space P and the map f appear redundent. Some of these conditions will be removed in Appendix 2.

(b) The most unsatisfactory feature of inequality (A.1) is dependence on the constant $n = \dim V$ rather than $m = \dim f^{-1}(p)$. Notice that the Almgren-Morse theory does provide lower bounds of $\mathrm{Vol}_m f$, which are independent of n. Unfortunately this theory does not fully apply (unlike the above filling argument) to non-Riemannian manifolds. However, some purely Riemannian problems require for their solution certain "slicing inequalities" like (A.1) with const $=$ const(m) for Finsler manifolds.

Examples. Let V_0 be a complete properly imbedded submanifold of dimension m in an $(n+1)$-dimensional Banach space L. We consider the intersections V_0 with the balls $B_x(R)$ of a fixed radius R in L, and want to find lower bounds for the following two volumes:

$$Vl_m(R) = Vl_m(R, V_0) = \sup_{x \in L} \operatorname{Vol}_m(V_0 \cap B_x(R)),$$
$$Vl_{m-1}(R) = Vl_{m-1}(R, V_0) = \sup_{x \in L} \operatorname{Vol}_{m-1}(V_0 \cap \partial B_x(R)),$$

for the boundary spheres $\partial B_x(R)$.

In particular we are interested in imbeddings into finite dimensional l^∞-spaces, $V_0 \subset l^\infty$, as these imbeddings approximate for $\dim l^\infty \to \infty$ the canonical imbedding $V_0 \subset L^\infty(V)$.

Question. Suppose that Fill Rad$(V_0 \subset L) > R$. Is it true that

$$Vl_m(R) \geqslant \text{const}\, R^m, \tag{A.2}$$
$$Vl_{m-1}(R) \geqslant \text{const}'\, R^{m-1}, \tag{A.3}$$

for some universal positive constants const $=$ const(m) and const$'$ $=$ const$'(m)$ (compare the question in (E_4'))?

One gets the positive answer to the above question for $L = \mathbf{R}^{n+1}$ (only!) by applying the Almgren-Morse theory as follows. Let P be an $(n-m)$-dimensional pseudomanifold in L with boundary ∂P, such that this boundary has nonzero linking with V_0 and such that $\operatorname{dist}(v, p) \geqslant R$ for all pairs of points $(v, p) \in V_0 \times \partial P$. Then we assign, to each point $p \in P$, a relative cycle V_p in the ball $B_0(R)$ by moving the intersection $B_p(R) \cap V_0$ to $B_0(R)$ by the vector $-p \in L$. Thus we "slice" the ball $B_0(R) \subset L$ into $(n-m)$-dimensional family of relative m-dimensional cycles $V_p \subset (B_0(R), \partial B_0(R))$, and then the Almgren-Morse theory gives a minimal relative cycle $V_{\min} \subset B_0(R)$, which has

$$\operatorname{Vol} B^m(R) \leqslant \operatorname{Vol} V_{min} \leqslant Vl_m(R),$$

for the m-dimensional Euclidean ball $B^m(R)$ of radius R. This proves (A.2) with const $= \operatorname{Vol} B^m(1)$, and the same argument yields (A.3) with const$'$ $= \operatorname{Vol} S^{m-1}(1)$, provided the submanifold V_0 is C^∞-generic or real analytic.

(G) Further questions. Let V be a closed Riemannian manifold of dimension n. Does there exist a closed (possibly contractible) geodesic γ in V, which has

$$\text{length}\,\gamma \leqslant \text{const}_n (\operatorname{Vol} V)^{1/n}?$$

The positive answer may be expected for surfaces V homeomorphic to S^2, as one is aided by conformal mappings $V \to S^2$.

The second question is due to Berger [11]. Let V be a 2-*essential* manifold of dimension n, that is, let V admit a map into the infinite dimensional complex projective space $f\colon V \to \mathbf{C}P^{\infty}$ such that $f_*[V] \neq 0$. Does the second systole of V satisfy

$$\operatorname{sys}_2 V \leqslant \operatorname{const}_n (\operatorname{Vol} V)^{2/n},$$

for some universal constant const_n?

Observe that the argument of §1.2 yields some information on another systolic invariant of V. Denote by $\alpha(V)$ the lower bound of the numbers $\varepsilon > 0$, for which there exists a surface V_0 with a Riemannian metric such that:

(1) V_0 is homeomorphic to S^2,

(2) $\operatorname{Rad}_1 V_0 \leqslant \varepsilon$,

(3) there exists a *noncontractible* distance-decreasing map of V_0 to V. (If we had used Area V_0 in place of $\operatorname{Rad}_1 V$, we would get a "spherical 2-systole" of V.)

Denote the length of the shortest geodesic in V by $l = l(V)$, and let R be the filling radius of V.

Proposition. *If the manifold V is* 2-*essential, and $l > 6R$, then*

$$\alpha(V) \leqslant 2R \leqslant \operatorname{const}_n (\operatorname{Vol} V)^{1/n},$$

and therefore

$$\min(l(V), \alpha(V)) \leqslant \operatorname{const}'_n (\operatorname{Vol} V)^{1/n}.$$

Proof. We subdivide a filling W of V into small simplices, and then retract the 1-skeleton of W to V as in §1.2.B. Since $l > 6R$, the boundary of every 2-simplex (as it is retracted to V) of W can be homotoped in V to a point by a family of curves of length $< l$. Thus we get a map of the 2-skeleton of W to V. The boundary of every 3-simplex Δ of W is "sliced" into curves of length $< l$ in V. Therefore every such boundary $\partial\Delta$ admits a metric, for which $\operatorname{Rad}_1 \partial\Delta \leqslant 2R$ and our (retraction) map $\partial\Delta \to V$ is distance-decreasing. Finally, as the manifold V is 2-essential, the map $\partial\Delta_0 \to V$ is noncontractible on the boundary $V_0 = \partial\Delta_0$ of some 3-simplex Δ_0.

Appendix 2. Filling inequalities in Finsler spaces

(A) On the topology of spanning chains. Our proof of the isoperimetric inequality for n-dimensional cycles in a Banach space L (see §§3.3, 4.2) applies to cycles z with arbitrary coefficients. However, the filling volume may depend on the chosen coefficient field. Nevertheless, the filling volume with *integral* coefficients of a connected oriented manifold V, $\dim V = n \geqslant 2$, provides a *universal* upper bound for all other filling volumes of V. Moreover, we have the following.

(A′) Proposition. *Consider an arbitrary smooth map* $f\colon V \to X$ *of the above manifold* V *into a Finsler manifold* X. *If the manifold* X *is contractible* ($X = L$, *for instance*), *then for every number* $\varepsilon > 0$ *there exists a cone* F *over* f (*that is, a map* $F\colon V \times [0,1] \to X$, *for which* $F \mid V \times 0 = f$ *and* $F \mid V \times 1 \equiv$ const.), *such that* $\operatorname{Vol} F \leqslant (1+\varepsilon)\operatorname{Fill Vol} f_*[V]$, *where* $f_*[V]$ *is the image of the fundamental cycle of* V *in some triangulation of* V, *and the volume of the map* F *is counted with geometric multiplicity.*

Proof. We assume without loss of generality that the cycle $f_*[V]$ in X is the fundamental class of an oriented sub-pseudomanifold V_* in X. We span V_* by an oriented pseudomanifold $W_*(\subset X)$ with boundary $\partial W_*(= V_*)$ such that

$$\operatorname{Vol} W_* \leqslant (1+\varepsilon)\operatorname{Fill Vol} V_*.$$

Then we consider a cylinder (map) $h\colon V_* \times [0,1] \to W_*$ with the following three properties:

(a) $h \mid V_* \times 0 = \mathrm{Id}$.

(b) The map $h_1 = h \mid V_* \times 1$ sends the pseudomanifold $V_* = V_* \times 1$ onto an n-dimensional subcomplex W' in W_*, i.e., $h_1\colon V_* \to W'$ for $\dim W' = n = \dim V^*$, and such that the map h_1 is homologous to zero, $h_1[V_*] = 0$.

(c) The map h is injective on the interior $V_* \times (0,1) \subset V_x \times [0,1]$, and so $\operatorname{Vol} h \leqslant \operatorname{Vol} W_*$.

As $\dim W' = n \geqslant 2$, we can attach some 2-handles to W' such that the resulting complex $W'' = W' +$ (the handles) also has dimension n and $\pi_1(W'') = 0$. Then there exists by the obstruction theory a homotopy of the map h_1 to the $(n-1)$-skeleton of W'':

$$h'\colon V_* \times [1,2] \to W'', \quad \text{for } h' \big| V_* \times 1 = h_1,$$

and such that the image $h^*(V_* \times 2)$ has dimension $\leqslant n-1$. Finally, we contract this image to a point $x \in X$ by a cone $h''\colon V_* \times [2,3]$ of zero $(n+1)$-dimensional volume, and take the composed homotopy $h \circ h' \circ h''$ for the cone F.

(A″) Remark. Proposition 2.2.A is an immediate corollary of Proposition (A′).

(A‴). The proposition (A′) does not apply (in fact it is false) for $V = S^1$ and also for nonorientable manifolds V of dimension $n \geqslant 2$. However, our proof of the isoperimetric inequality in §§3.3 and 4.2 is consistent with the argument in [36], and so it implies the following.

Theorem. *Let* W *be an arbitrary* $(n+1)$*-dimensional polyhedron* $n \geqslant 1$, *and let* V *be an* n*-dimensional subpolyhedron in* W. *Then an arbitrary piecewise* *smooth map f: $V \to L$ of V into a Banach space extends to a piecewise smooth map F: $W \to L$, such that*

$$\operatorname{Vol} F \leqslant n^{2n}(\operatorname{Vol} f)^{(n+1)/n},$$

where the $(n+1)$-dimensional volume of F (as well as the n-dimensional volume of f) is counted with geometric multiplicity.

As this theorem is not used in the sequel, we leave the proof to the reader.

(B) Contraction invariants of metric spaces. The results of the previous section show that there is little interraction between the volume and the topology of the filling "manifold" W. However, the topology of W becomes more relevant if we take into account the filling radius as well as the filling volume.

Recall that a continuous map between two metric spaces is said to be *k-contractible* if it is homotopic to a k-degenerate map (see Appendix 1). For an arbitrary compact subspace V in a metric space X we define *the contractibility radius* $\operatorname{Cont}_k \operatorname{Rad}(V \subset X)$ to be the lower bound of the numbers $\varepsilon > 0$, for which the inclusion map of V into its ε-neighborhood $U_\varepsilon(V) \supset V$ is a k-contractible map. Using the canonical embedding $V \subset L^\infty(V)$ we can define the contractibility radius of a (non-embedded) metric space V as follows:

$$\operatorname{Cont}_k \operatorname{Rad} V \overset{\text{def}}{=} \operatorname{Cont}_k \operatorname{Rad}(V \subset L^\infty V).$$

The following properties of this radius are immediate from the definition.

(a) $\infty \geqslant \operatorname{Cont}_0 \operatorname{Rad}(V) \geqslant \operatorname{Cont}_1 \operatorname{Rad} V \geqslant \cdots \geqslant \operatorname{cont}_n \operatorname{Rad} V = 0$, for $n = \dim V$.

(b) $\operatorname{Cont}_k \operatorname{Rad} V \leqslant \operatorname{Rad}_k V$, for all $k = 0, 1, \ldots$.

(c) If V is a closed n-dimensional manifold, then

$$\operatorname{Cont}_{n-1} \operatorname{Rad} V \geqslant \operatorname{Fill} \operatorname{Rad} V.$$

Furthermore, by the obstruction theory (compare (A)) one obtains the equality

$$\operatorname{Cont}_{n-1} \operatorname{Rad} V = \operatorname{Fill} \operatorname{Rad} V,$$

for *simply connected* manifolds V. But if $\pi_1(V) \neq 0$, then an upper bound of the radius $\operatorname{Cont}_{n-1} \operatorname{Rad} V$ cannot be obtained by using the filling radius or (and) the filling volume. However, we shall see below how a minor modification of our filling technique of §3.4 yields the following.

($\mathbf{B}_1$) Theorem. *An arbitrary compact n-dimensional polyhedron V with a piecewise Riemannian (or Finsler) metric has*

$$\operatorname{Cont}_{n-1} \operatorname{Rad} V \leqslant \operatorname{const}_n (\operatorname{Vol} V)^{1/n},$$

for some universal constant $\operatorname{const}_n > 0$.

Next we say that an n-dimensional polyhedron V is "essential" if there exists a continuous map f of V into some $K(\Pi, 1)$-space such that f is *not* $(n-1)$-contractible. Observe that every essential manifold V is also "essential", but the converse is hardly true.

(B_1') Isosystolic inequality for polyhedra (compare §6.7). *Every compact "essential" polyhedron has*

$$\operatorname{sys}_1 V \leqslant 6\operatorname{const}_n(\operatorname{Vol} V)^{1/n}.$$

Indeed, as in §1.2 one obtains the inequality

$$\operatorname{sys}_1 V \leqslant 6\operatorname{Cont}_{n-1}\operatorname{Rad} V.$$

A polyhedron W is said to be *k-contractible* if the identity map is k-contractible. If V is a subpolyhedron in a space X, then we call a polyhedron W in X a *k-contraction* of V if W is k-contractible and contains V, i.e.,

$$V \subset W \subset X.$$

We denote the upper bound $\sup_{w \in W} \operatorname{dist}(w, V)$ by $\operatorname{Rad} W = \operatorname{Rad}(W, V)$.

Finally, we denote by $\operatorname{Cont}_{n-1} V$, for an n-dimensional polyhedron V with an arbitrary metric, the lower bound of those $\varepsilon > 0$, for which there exists a k-contraction W of the canonical imbedding $V \subset L^\infty(V)$, that is,

$$V \subset W \subset L^\infty(V),$$

such that $\dim W = n + 1$, $\operatorname{Rad} W < \varepsilon$, and the $(n+1)$-dimensional volume of W satisfies

$$\operatorname{Vol} W \leqslant \varepsilon^{n+1}.$$

Theorem (B_1) can be strengthened (see (C) below) as follows.

(B_2) Theorem. *If V satisfies the assumptions of Theorem (B_1), then*

$$\operatorname{Cont}_{n-1} V \leqslant \operatorname{const}_n'(\operatorname{Vol} V)^{1/n}.$$

(B_2') Corollary. *Let f be a piecewise real analytic (for instance piecewise linear) map of V onto a $(m-n)$-dimensional polyhedron P. Then*

$$\operatorname{Cont}_{n-1}\operatorname{Rad} V \leqslant \operatorname{const}_n'' \operatorname{Vol}_m f.$$

In particular, if V is an n-dimensional manifold, then

$$\operatorname{Fill}\operatorname{Rad} V \leqslant \operatorname{const}_n'' \operatorname{Vol} f.$$

(Compare (F) in App. 1.)

Proof. The argument of (F_2'') in Appendix 1 goes along with the following lemmas

(B_3) *Let A be a k-contractible space, and B a $(k-1)$-contractible subspace in A. Then every $(k-1)$-contracting homotopy of B, that is, a homotopy h: $B \times [0,1] \to B$, for which $h \mid B \times 0 = \operatorname{Id}$ and the map $h \mid B \times I$ is $(k-1)$-degenerate, extends to a k-contracting homotopy of A.*

The proof is straightforward.

(B_3') *Let f: $W \to P$ be a continuous map between polyhedra such that the pullback of every k-dimensional simplex in P is a $(k+m-1)$-contractible subpolyhedron in W for all $k = 0,\cdots,n-m = \dim P$. Then the polyhedron W is $(n-1)$-contractible.*

Proof. Use the previous lemma and the induction by skeletons of the polyhedron P.

(B_4) Example. Let V be a closed essential manifold, and let f: $V \to \mathbf{R}^{n-m}$ be a real analytic or generic C^∞-map. Then there is a *connected component* V_0 of the pullback of some point $p \in P$:

$$V_0 \subset f^{-1}(p) \subset V,$$

which has

$$\text{(A.4)} \qquad \operatorname{Vol}_m V_0 \geqslant \delta(\operatorname{sys}_1 V)^m$$

for some universal positive constant $\delta = \delta(n) > 0$.

Proof. Apply Corollary (B_2') to the associated connected map $\bar{f}$: $V \to P = \tilde{\mathbf{R}}^{n-m}$, and use appropriate piecewise linear structures in V and in P. (Compare (E_1) and (E_1') of Appendix 1.)

Observe that (A.4) for $m = n$ reduces to the isosystolic inequality $\operatorname{Vol} V \geqslant \delta(\operatorname{sys}_1 V)^n$.

It is unclear if inequality (A.4) holds with a constant $\delta = \delta(m) > 0$.

(C) The proof of Theorem (B_2). We slightly generalize the above definitions by considering *singular* polyhedra V in X, which are maps f: $V \to X$. Then a k-contraction of V by definition is an extension of f to a map of a k-contractible polyhedron F: $W \to X$ for $W \supset V$, where $F | V = f$. We denote by $\operatorname{Vol} V = \operatorname{Vol}_n V$, for $n = \dim V$, the volume of the map f counted with geometric multiplicity. The same notation applies to all other singular polyhedra in question. We put $\operatorname{Rad} W = \sup_{w \in W} \operatorname{dist}(f(V), w)$, and then define the (total) contraction radius of V, $\operatorname{Cont}_{n-1}(V \to X)$, as the smallest ε for which there exists an $(n-1)$-contraction W of V such that the quantities $\operatorname{Rad} W$ and $(\operatorname{Vol}_{n+1} W)^{1/(n+1)}$ are bounded from above by ε.

Observe that these modifications are not needed for polyhedra in spaces X of large dimension ($\geqslant 2n+2$), as all maps can be made injective by small perturbations.

Now we proceed with the proof of Theorem (B_2) by indicating the required modification of the argument in §3.4.

(C_1) *Every singular n-dimensional polyhedron V in $\mathbf{R}^N$ has*

$$\operatorname{Cont}_{n-1}(V \to \mathbf{R}^N) \leqslant C_N(\operatorname{Vol} V)^{1/n}.$$

Indeed, the Federer-Fleming proof (see §3.2) applies.

(C_1') Let X be a compact (possibly with boundary) Finsler manifold. Then there exist positive constants $\varepsilon = \varepsilon(X) > 0$ and $C = C(X) > 0$ such that every singular polyhedron V in X of dimension n and $\operatorname{Vol} V \leqslant \varepsilon$ satisfies

$$\operatorname{Cont}_{n-1}(V \subset X) \leqslant C(\operatorname{Vol} V)^{1/n}.$$

Proof. Imbed X to $\mathbf{R}^N$, contract V by some polyhedron W in $\mathbf{R}^N$ according to (C_1) and then normally project this W to $X \subset \mathbf{R}^N$.

(C_2) Decomposition of polyhedra. Let a singular polyhedron V in some space X be decomposed into the union of two subpolyhedra, i.e. $V = V_1' \cup V_2'$, such that the intersection $V_0' = V_1' \cap V_2'$ has dimension $n - 1$ for $n = \dim V$. Let W_0 be a $(n-2)$-contraction of V_0' in X. We put $V_1 = V_1' \cup W_0$ and $V_2 = V_2' \cup W_0$, and we say that V is decomposed into the "sum": $V = V_1 + V_2$.

(C_2') *If W_1 is an $(n-1)$-contraction of V_1, and W_2 is an $(n-1)$-contraction of V_2, then the union $W_1 \cup W_2 \to X$ is an $(n-1)$-contraction of V.*

This is immediate from (B_3).

Remark. The above "sum" by no means is an associate operation. However, we shall omit brackets, and we even shall write ΣV_i for finite "sums" $V_i + V_2 + \cdots$.

(C_3) *Let X be a compact Finsler manifold. Suppose that every $(n-1)$-dimenional singular polyhedron V' in X of volume $< (\delta_{n-1})^{n-1}$ for some fixed number $\delta_{n-1} > 0$, admits an $(n-2)$-contraction W' such that*

$$\operatorname{Vol} W' \leqslant C_{n-1}(\operatorname{Vol} V')^{n/(n-1)}, \tag{A.5}$$

for some fixed constant C_{n-1}. Then there exist two positive constants $\delta > 0$ and $D > 0$ depending only on δ_{n-1} and C_{n-1} such that every n-dimensional singular polyhedron V in X of volume $\leqslant \delta^n$ admits a "sum" decomposition, $V = \Sigma_\mu V_\mu + \Sigma_\nu V_\nu$, with the following three properties:

(1) $\Sigma_\mu (\operatorname{Vol} V_\mu)^{(n+1)/n} + \Sigma_\nu (\operatorname{Vol} V_\nu)^{(n+1)/n} \leqslant (\operatorname{Vol} V)^{(n+1)/n}$

(2) *Every singular polyhedron V_μ is D-round, i.e.,*

$$\operatorname{Diam} V_\mu \leqslant D(\operatorname{Vol} V_\mu)^{1/n},$$

where the diameter of a singular polyhedron by definition is the diameter of its image in X.

(3) $\Sigma_\nu (\operatorname{Vol} V_\nu)^{(n+1)/n} \leqslant \varepsilon$, *where $\varepsilon > 0$ is an arbitrarily chosen small positive number.*

Proof. We introduce (compare §3.4) the weighted volume, Weight V, as the lower bound of the sums $\Sigma_i (\operatorname{Vol} V_i)^{(n+1)/n}$ over all finite "sum" decompositions, $V = \Sigma_i V_i$. Then we take some decomposition $V = \Sigma_i V_i$, for which the sum $\Sigma_i (\operatorname{Vol}_i)^{(n+1)/n}$ is very close to Weight V. Then "almost all" polyhedra V_i are "essentially round"; namely, a polyhedron $V_\mu \to X$ is called *ε-essentially*

D-round if there is a ball $B = B_x(R)$ in X of radius $R \leqslant \frac{1}{3}D(\operatorname{Vol} V_\mu)^{1/n}$ such that the volume of the part of V_μ in B satisfies

$$\operatorname{Vol}(B \cap V_\mu) \geqslant \left(1 - \frac{\varepsilon}{2}\right)\operatorname{Vol} V_\mu .$$

Now if some polyhedron V_i (whose volume is small compared to δ_{n-1}^{n-1}) is not "round", then it can be decomposed further with a substantial decrease of its weighted volume. This is done as in Lemma 3.4.A by intersecting this V_i with some level of a distance function $d\colon X \to \mathbf{R}$, and then by using a small $(n-2)$-contraction W_i' of this intersection. It follows that no ε-essentially D-round polyhedra V_ν among V_i, for some constant $D = D(C_{n-1}, \delta_{n-1}^{-1})$ and for $\varepsilon \to 0$ as $\Sigma_i (\operatorname{Vol} V_i)^{(n+1)/n}$ approaches Weight V, have $\Sigma_\nu (\operatorname{Vol} V_\nu)^{(n+1)/n} \to 0$.

Next every essentially round polyhedron V_i can be made round: intersect V_i by the boundary of some small ball B, which contains almost all of V_i, and then $(n-2)$-contract this intersection according to (C_1'). Thus V_i is decomposed into a round part (in B) and a negligible term which then goes to the sum $\Sigma_\nu V_\nu$. (Compare §3.4.) q.e.d.

Lemma (C_3) allows one to derive the isoperimetric inequality for *contractions* in X from an appropriate "cone inequality". But this is not so useful as in the case of filling *chains*, since the filling radius is not controlled any more by the filling volume. However, one can use Lemma (C_3) in the following less straightforward way in order to control the distances of the polyhedra V_μ and V_μ from V and thus to estimate $\operatorname{Cont}_{n-1}(V \to X)$.

We say that a Finsler manifold X satisfies inequality $\operatorname{Is}_n = \operatorname{Is}_n(C, \delta^{-1})$, if every n-dimensional polyhedron $V \to X$ of volume $\leqslant \delta^n$ can be $(n-1)$-contracted by a polyhedron W which has

$$(\operatorname{Is}_n) \qquad \operatorname{Vol} W \leqslant C(\operatorname{Vol} V)^{(n+1)/n}.$$

We say that X satisfies the inequality $\operatorname{Contr}_n = \operatorname{Contr}_n(C, \delta^{-1})$, if the above V has

$$(\operatorname{Contr}_n) \qquad \operatorname{Cont}_{n-1}(V) \leqslant C(\operatorname{Vol})^{1/n}$$

This amounts (up to an irrelevant descrepency between C and $C^{1/(n+1)}$) to the existence of a W which satisfies Is_n as well as the inequality

$$(\operatorname{Rad}_n) \qquad \operatorname{Rad} W \leqslant C(\operatorname{vol} V)^{1/n}$$

Finally, the manifold X is said to satisfy $\operatorname{Cone}_n = \operatorname{Cone}_n(C, \delta^{-1}, D)$, for $D \geqslant 1$, if inequality Contr_n only holds for those polyhedra V of volume $\leqslant \delta^n$, which are *d-round*:

$$\operatorname{Diam} V \leqslant D(\operatorname{Vol} V)^{1/n}.$$

Lemmas (C_3) and (C_1) show that ($\mathrm{Is}_{n-1} + \mathrm{Cone}_n$) $\Rightarrow \mathrm{Is}_n$ for compact manifolds X. More explicitly,

$$\left(\mathrm{Is}_{n-1}\left(C_{n-1}, \delta_{n-1}^{-1}\right) + \mathrm{Cone}_n\left(C_n, \delta_n^{-1} D_n\right)\right) \Rightarrow \mathrm{Is}_n\left(\bar{C}_n, \bar{\delta}_n^{-1}\right),$$

provided the constant D_n is sufficiently large, i.e., $D_n \geqslant \Delta = \Delta(n, C_{n-1}, \delta_{n-1}^{-1})$, and then the constants $\bar{C}_n$ and $\bar{\delta}_n^{-1}$ depend only on C_{n-1}, δ_{n-1}^{-1}, C_n and δ_n^{-1}.

Next by induction we conclude that the inequalities Cone_k for $k = 1, \cdots, n$, imply the inequalities Is_k, i.e.,

$$\sum_{k=1}^{n} \mathrm{Cone}_k \Rightarrow \sum_{k=1}^{n} \mathrm{Is}_k,$$

with obvious rule for constants: each constant D_k must be sufficiently large compared to the constants C_i and δ_i^{-1} of Cone_i for $i < k$, and then the constants of $\Sigma_1^n \mathrm{Is}_k$ depend only on n, the constants C_k and δ_k^{-1} of $\Sigma_k \mathrm{Cone}_k$.

(C_4) Theorem. *The cone inequalities imply the contraction inequalities, i.e.,*

$$\sum_1^n \mathrm{Cone}_k \Rightarrow \sum_1^n \mathrm{Contr}_k,$$

for an arbitrary complete Finsler manifold X.

Proof. Let V be a compact n-dimensional singular polyhedron in X, whose volume is small compared to the constants C_k and δ_k^{-1} in $\Sigma_{k=1}^n \mathrm{Cone}_k$. Take a compact submanifold $Y \subset X$ with smooth boundary which contains the ρ-neighborhood of the image of V in X such that $\rho \geqslant C(\mathrm{Vol}\, V)^{1/n}$ for some constant C which is sufficiently large compared to the constants in C_k and δ_k in $\Sigma_k \mathrm{Cone}_k$.

Let us introduce a new metric in the interior of Y by multiplying the original Finsler metric by the following function $f = f(y)$, $y \in \mathrm{Int}\, Y$:

$$f(y) = \max\left(1, \rho\left[\mathrm{dist}(y, \partial Y)\right]^{-1}\right).$$

The new length of each curve γ in $\tilde{Y} = \mathrm{Int}\, Y$ by definition is the integral $\int_\gamma f\, d\gamma$.

The function $\log f(y)$ is Lipschitz relative to the new metric with the Lipschitz constant $\mathrm{Dil} \log f \leqslant \rho^{-1}$. It follows that every cone inequality $\mathrm{Cone}_k = \mathrm{Cone}_k(C_k, \delta_k^{-1}, D_k)$ in X yields the cone inequality in $\tilde{Y} = (\tilde{Y}$, new metric); namely, $\mathrm{Cone}_k(\tilde{C}_k, \tilde{\delta}_k^{-1}, \tilde{D}_k)$ holds in $\tilde{Y}$ with the new metric for

$$\tilde{\delta}_k = \min(\delta_k, \rho/(2C_k + 2)),$$

$$\tilde{C}_k = 4C_k, \quad D_k = (\rho \log D_k)/\tilde{\delta}_k.$$

Furthermore, the same cone inequality holds in the *compact* region $\tilde{Y}_\varepsilon = f^{-1}(1, \varepsilon^{-1})$ in $\tilde{Y}$ provided the number $\varepsilon > 0$ is sufficiently small. Indeed, the normal projection of the set $\tilde{Y} - \tilde{Y}_\varepsilon$ to the hypersurface $f^{-1}(\varepsilon^{-1})$ in $\tilde{Y}$ is distance-decreasing in the new metric.

Now the cone inequalities in $\tilde{Y}_\varepsilon$ imply the isoperimetric inequalities. In particular we get an $(n-1)$-contraction W of V in $\tilde{Y}$, which has a controlled volume, and this contraction has, relative to the old metric,

$$\operatorname{Rad} W \leqslant \sup_{y \in Y} \operatorname{dist}(y, V) = \rho.$$

Therefore for some choice of $\rho = C(\operatorname{Vol} V)^{1/n}$ with a large controlled constant C, we get the required bound on $\operatorname{Cont}_{n-1} V$.

(C_4') Additional remarks and corollaries. The above localization trick (old metric) → (new metric) also works in the context of §3.4. and so we extend the results of that section to complete *noncompact* manifolds. In fact, our argument equally applies to arbitrary polyhedra X with complete piecewise Finsler metrics.

(C_5) The conclusion of the proof of Theorem (B_2). We approximate the canonical imbedding $V \subset L^\infty(V)$ by some imbedding into a finite dimensional subspace L in $L^\infty(V)$. As every Banach space satisfies the cone inequalities, Theorem (C_4) applies.

Appendix 3. Hausdorff Convergence

Let A be a subspace in a metric space X. We denote the function $d_A(x) = \operatorname{dist}(A, x)$ by $d_A \in L^\infty(X)$. The Hausdorff distance between subspaces A and B in X is

$$\operatorname{dist}_X(A, B) \overset{\text{def}}{=} \operatorname{dist}(d_A, d_B) = \|d_A - d_B\|_{L^\infty}.$$

For two abstract metric spaces A and B we then define the (abstract) Hausdorff distance Haus dist(A, B) by first considering all possible metrics spaces $X = (A \cup B, \rho)$ whose metric ρ agrees with the given metrics in A and B, and then by putting

$$\operatorname{Haus\,dist}(A, B) = \inf_X \operatorname{dist}_X(A, B).$$

This Hausdorff distance is, in fact, a metric on the set of the isometry classes of *compact* metric spaces (see [7]).

A family of metric spaces $\{A_\mu\}$, $\mu \in M$, admits a Hausdorff convergent subsequence if and only if the spaces A_μ are *uniformly compact*, i.e., if and only if the diameters of A_μ are uniformly bounded:

$$\operatorname{Diam} A_\mu \leqslant \text{const} < \infty \quad \text{for all } \mu \in M,$$

and there exists a function $N(\varepsilon)$, for $0 < \varepsilon < \infty$, such that every space A admits a cover by at most $N(\varepsilon)$ balls of radius ε. This condition is satisfied, for example, if every space A_μ is provided with a measure such that every ε-ball B_ε in each space A_μ satisfies

$$\operatorname{mes} B_\varepsilon \geqslant \operatorname{const}_\varepsilon \operatorname{mes} A_\mu,$$

for all $\varepsilon > 0$ (see **[37]** for the proofs and further properties of the Hausdorff distance).

Notice finally that abstract Hausdorff convergence $A_i \to A_\infty$ can be reduced to the ordinary Hausdorff convergence of subsets; namely, there are some isometric imbeddings of the spaces A_i and A into a compact metric space X such that $\operatorname{dist}_X(A_i, A_\infty) \to 0$ for $i \to \infty$. Also observe that a limit of *length spaces*, for which $\operatorname{dist}(a, b) = \inf(\text{lengths of curves between } a \text{ and } b)$, is also a length space.

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